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On the Eilenberg-MacLane invariants of loop spaces.

By Haruo SUZUKI

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1. Let X be a simply connected topological space and let E be the space of all paths in X starting from a fixed point $x_0 \in X$, topologized by compact-open topology. Then E is contractible, and with the projection $\rho: E \rightarrow X$ which associates each path to its terminal point, (E, ρ, X) is a fiber space in the sense of Serre [1], where the fiber at x_0 is the loop space Ω_X of X. It is well known that we have $\pi_i(X) \approx \pi_{i-1}(\Omega_X)$, $i=2, 3, \cdots$.

Fixing integers p, q such that $2<p<q$, we assume in the following that $\pi_i(X)=0$ for $p \neq i<q$ and put $\pi_p(X)=\pi_p$, $\pi_q(X)=\pi_q$. Then $\pi_i(\Omega_X)=0$ for $p \neq i+1<q$, and $\pi_{p-1}(\Omega_X) \approx \pi_p$, $\pi_{q-1}(\Omega_X) \approx \pi_q$. We shall put $\pi_{p-1}(\Omega_X)=\pi_{p-1}$, $\pi_{q-1}(\Omega_X)=\pi_{q-1}$, and consider these groups with the canonical isomorphisms $\pi_p \approx \pi_{p-1}$, $\pi_q \approx \pi_{q-1}$.

Now, the spaces X and Ω_X determine the Eilenberg-MacLane invariants $\boldsymbol{k}_p^{q+1}(X) \in H^{q+1}(\pi_p, p, \pi_q)$ and $\boldsymbol{k}_{p-1}^q(\Omega_X) \in H^q(\pi_{p-1}, p-1, \pi_{q-1})$ respectively. As will be shown, the latter invariant $\boldsymbol{k}_{p-1}^q(\Omega_X)$ is the image of the former $\boldsymbol{k}_p^{q+1}(X)$ under the suspension homomorphism S of the cohomology groups (Theorem 2 below). Therefore, if we associate to any system $(\pi, \pi', \boldsymbol{k}_p^{q+1})$ consisting of abelian groups π, π' and an element $\boldsymbol{k}_p^{q+1}$ in $H^{q+1}(\pi, p, \pi')$ the system $S^*(\pi, \pi', \boldsymbol{k}_p^{q+1})=(\pi, \pi', S\boldsymbol{k}_p^{q+1})$, then the correspondence S^* has a geometrical meaning.

If q is sufficiently small, we can define the inverse of this operation. When X is a CW-complex the homotopy type of X is determined by that of Ω_X[1] (see §3 Cor. 4 below). There is also an analogous relation about invariants of J. H. C. Whitehead which we shall show for standard complexes and standard loop spaces [5].

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1) Conversely, in arcwise connected spaces, homotopy types of loop spaces are determined by those of original spaces (see §5 below).