Journal of the Mathematical Society of Japan Vol. 7, Supplement, December, 1955

Theory of arithmetic linear transformations and its application to an elementary proof of Dirichlet's theorem.

By Koichi YAMAMOTO

(Received Nov. 25, 1955)

Introduction.

A. Selberg [8] proved Dirichlet's theorem about the primes in an infinite arithmetic progression in an elementary way. His basic idea was in the introduction of a new kind of asymptotic formulas, a typical one being the so-called Selberg's asymptotic equality. His method, however, is not entirely different from those used in classical proofs since Dirichlet, and this fact is explained in the present paper by a principle, which we would call theory of arithmetic linear transformations.

This principle was, in its essence, perceived by many authors, notably by Möbius, Glaisher (Cf. [2, Chapt. XIX]), Landau, Selberg and others (Cf. [7, 10]), but seems not fully recognized.

It is observed that any of the existing proofs of Dirichlet's theorem consists of the two steps. The first, and the formal part is the derivation of the theorem from the fact that $\beta(\chi)=\sum_{n=1}^{\infty}\chi(n)/n\neq 0$ for any non-principal real character χ mod k. And the second, more conceptual part is the proof of this fact.

As for the first part, the classical proofs make use of Dirichlet L-fuuctions associated with characters χ mod k (which are generating functions of χ in the form of Dirichlet series), whereas Selberg replaces it by Selberg's asymptotic equality. In the idea of the proof, the classical method is more elementary, since it aims to generalize the well-known Mertens-Polignac formula $\sum_{p\leqq x}\log p/p=\log x+O(1)$, whereas Selberg uses a more complicated sum. Here we can proceed along the classical line without resorting to L-functions.