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# Hopf's ergodic theorem on Fuchsian groups.

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## 1. Hopf's theorem.

Let a non-euclidean metric in $|z|<1$ be defined by

$$ds=\frac{2|dz|}{1-|z|^2}, \qquad d\sigma=\frac{4dxdy}{(1-|z|^2)^2}\ (z=x+iy). \tag{1}$$

Let $\eta_1=e^{i\theta}$, $\eta_2=e^{i\varphi}$ be two points on $|z|=1$, then the pair $(\eta_1, \eta_2)$ can be considered as a point of a torus $\Theta: 0\leqq\theta\leqq 2\pi,\ 0\leqq\varphi\leqq 2\pi$. For a measurable set $E$ on $\Theta$, we define its measure by

$$\mu(E)=\iint_E d\theta d\varphi, \quad \text{so that} \quad \mu(\Theta)=4\pi^2. \tag{2}$$

Let $G$ be a Fuchsian group of linear transformations, which make $|z|<1$ invariant and $S_\nu$ be any substitution of $G$ and

$$T_\nu: \eta_1'=S_\nu(\eta_1), \quad \eta_2'=S_\nu(\eta_2),$$

then the totality of $T_\nu$ consititutes a group $\mathfrak{G}=G\times G$. Hopf[1] proved

THEOREM 1. *Let $D_0$ be the fundamental domain of $G$. If $\sigma(D_0)<\infty$, then there exists no measurable set $E$ on $\Theta$, which is invariant by $\mathfrak{G}$ and $0<\mu(E)<4\pi^2$, so that if $\mu(E)>0$, then $\mu(E)=4\pi^2$.*

In the former paper[2], I gave another proof of the theorem. I could simplify my former proof a little, which we shall show in § 3 and as an application of the theorem, we shall prove an analogue of Weyl's theorem on uniform distribution for Fuchsian groups in § 5.

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1) E. Hopf: Fuchsian groups and ergodic theory. Trans. Amer. Math. Soc. 39 (1936). Ergodentheorie. Berlin (1937).

2) M. Tsuji: On Hopf's ergodic theorem. Proc. Imp. Acad. (1944). Jap. Journ. Math. 19 (1945).