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On the fundamental conjecture of GLC I.

By Gaisi TAKEUTI

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G. Gentzen has founded in his well known paper [1] a logic calculus LK, and proved the remarkable result that every provable sequence in LK is provable without cut, by means of which he could establish the consistency of the number theory [2]. The author has generalized in his former paper [4] Gentzen's LK to a logical system GLC (Generalized Logic Calculus), containing a subsystem G^1LC, which latter contains LK. The proposition "Every provable sequence in GLC (resp. G^1LC) is provable without cut" was called the fundamental conjecture of GLC (resp. G^1LC), and it was shown that from this conjecture would follow the consistency of the analysis (resp. of the theory of real numbers).

We shall prove in this paper the following theorem, which may be regarded as a special case of the fundamental conjecture of G^1LC.

THEOREM. *Let* $\mathfrak{P}$ *be a proof-figure of a sequence* $\mathfrak{S}$ *in* G^1LC. *Assume that no beginning sequence of* $\mathfrak{P}$ *contains logical symbols, and that* $\mathfrak{P}$ *has no inference-figure* $\forall, \exists$ *on variable of height* 1 *of the forms described below:*

$$\frac{F(V), \Gamma \rightarrow \Delta}{\forall \varphi F(\varphi), \Gamma \rightarrow \Delta}, \qquad \frac{\Gamma \rightarrow \Delta, F(V)}{\Gamma \rightarrow \Delta, \exists \varphi F(\varphi)},$$

where $F(\alpha)$ *has a proper* $\forall$ *or* $\exists$ *on variable of height* 1. *Then* $\mathfrak{S}$ *is provable without cut.*

From this theorem follows the consistency of the theory of natural numbers. In fact the mathematical induction is formalized in GL^1C by the following inference-figures. (See § 4 in our former paper [4])

$$\frac{\Gamma \rightarrow \Delta, \alpha[0] \wedge \forall x(\alpha[x] \vdash \alpha[x']) \vdash \alpha[T]}{\Gamma \rightarrow \Delta, \forall \varphi(\varphi[0] \wedge \forall x(\varphi[x] \vdash \varphi[x']) \vdash \varphi[T])},$$

$$\frac{A(0) \wedge \forall x(A(x) \vdash A(x')) \vdash A(T), \Gamma \rightarrow \Delta}{\forall \varphi(\varphi[0] \wedge \forall x(\varphi[x] \vdash \varphi[x']) \vdash \varphi[T]), \Gamma \rightarrow \Delta}.$$