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Function of U-class and its applications.

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1. Function of U-class.

Let $w=f(z)$ be regular and $|f(z)|<1$ in $|z|<1$, then by Fatou's theorem, $\lim_{z\to e^{i\theta}} f(z)=f(e^{i\theta})$ exists almost everywhere on $|z|=1$, when $z\to e^{i\theta}$ from the inside of any Stolz domain, whose vertex is at $e^{i\theta}$. If $|f(e^{i\theta})|=1$ almost everywhere, we say with Seidel[1] that $f(z)$ belongs to U-class and denote $f(z)\in U$. If $(f(z)-a)/\rho\in U$, we write $f(z)\in U_{\rho}(a)$. Functions of U-class play an important rôle in several problems. In this paper, we shall show some applications of them. In this paper, "capacity" means "logarithmic capacity" and $\gamma(E)$ denotes the capacity of E.

LEMMA 1.[2] (*Extension of Löwner's theorem*). *Let $w=f(z)$ be regular and $|f(z)|<1$ in $|z|<1, f(0)=0$. Let E be the set of $e^{i\theta}$ on $|z|=1$, such that $|f(e^{i\theta})|=1$ and E^* be the set $f(e^{i\theta})$ $(e^{i\theta}\in E)$ on $|w|=1$. Then E and E^* are measurable and $mE^*\geqq mE$.*

LEMMA 2. *If $f(z)\in U$, then $f(z)$ takes any value of $|w|<1$ at least once, except a set of capacity zero.*

PROOF. Let E be the set of a $(|a|<1)$, such that $f(z)\neq a$ in $|z|<1$ and suppose that $\gamma(E)>0$, then by taking a suitable closed sub-set, we may assume that E is a closed set, contained entirely in $|w|<1$. Let D be the domain, which is bounded by E and $|w|=1$. We solve the Dirichlet problem for D, with the boundary value 1 on E and 0 on $|w|=1$, and let $u(w)$ be its solution, then since $\gamma(E)>0$, E contains a regular point of Dirichlet problem, so that $u(w)\not\equiv 0$. If we put $u(f(z))=v(z)$, then $v(z)$ is a bounded harmonic function in $|z|<1$.

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1) W. Seidel: On the distribution of values of bounded analytic functions. Trans. Amer. Math. Soc. 36 (1934).

2) M. Tsuji: On an extension of Löwner's theorm. Proc. Imp. Acad. 18 (1942). The special case, where $f(z)$ is schlicht in $|z|<1$, is proved by Y. Kawakami: On an extension of Löwner's lemma. Jap. Journ. Math. 17 (1941).