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On a positive harmonic function in a half-plane.

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THEOREM 1. *Let $u(z)=u(x+iy)$ be harmonic and $u>0$ for $x>0$. Let C be a Jordan arc, contained in the half-plane $x>0$, which ends at $z=0$ and is contained in a Stolz domain, whose vertex is at $z=0$. If $u(z)$ is bounded on C, then $u(z)$ is bounded in a sector Δ: $|z|\leqq 1$, $|\arg z|\leqq\varphi_0<\frac{\pi}{2}$.*

PROOF. Since $u(z)>0$ for $x>0$, $u(z)$ can be expressed by

$$u(z)=\int_{-\infty}^{\infty}\frac{xd\chi(t)}{x^2+(y-t)^2}+cx, \qquad c\geqq 0, \tag{1}$$

where $\chi(t)$ is an increasing function of t, such that $\chi(0)=0$.[1)]

From (1),

$$\int_{|t|\geqq 1}\frac{d\chi(t)}{t^2}<\infty. \tag{2}$$

Let $0<u(z)\leqq M$ on C and $z=x+iy$ lie on C, then $|y|\leqq k_0x$ (k_0=const.), so that

$$M\geqq u(z)-cx\geqq\int_{-x}^{x}\frac{xd\chi(t)}{x^2+(|y|+|t|)^2}\geqq\int_{-x}^{x}\frac{d\chi(t)}{x(1+(k_0+1)^2)}$$
$$=\frac{\chi(x)-\chi(-x)}{x(1+(k_0+1)^2)},$$

hence

$$|\chi(t)|\leqq K|t|, \qquad |t|\leqq 1, \qquad K=(1+(k_0+1)^2)M. \tag{3}$$

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1) A. Dinghas: Über das Phragmén-Lindelöfsche Prinzip und den Julia-Carathéodoryschen Satz. Sitzungsber. Preuss. Akad. Wiss. (1938). M. Tsuji: On a positive harmonic function in a half-plane. Jap. Journ. Math. 15 (1939).