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Logarithmic order of free distributive lattice

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1.—Introduction.—The problem to determine the order $f(n)$ of the free distributive lattice $FD(n)$ generated by n symbols $\gamma_1, \cdots, \gamma_n$ was first proposed by Dedekind, but very little is known about this number [1, p. 146]. Only the first six values of $f(n)$ are computed, and enumerations of further $f(n)$ appear to lie beyond the scope of any reasonable methods known today. It might, however, be pointed out that Morgan Ward, who found $f(6)$ by the help of computing machines, stated [2] an asymptotic relation

$$\log_2 \log_2 f(n) \sim n$$

and that the present author proved in a previous note [3] that

$$f(n) \equiv 0 \pmod 2 \qquad \text{if} \qquad n \equiv 0 \pmod 2).$$

An inspection of numerical results $f(n)$, $n \leqq 6$ suggests strongly the following asymptotic equivalence

$$(*) \qquad \log_2 f(n) \sim \sqrt{\frac{2}{\pi}}\, 2^n n^{-\frac{1}{2}}.$$

The author cannot prove or disprove this interesting relation, but he proves in the present paper that

$$\sqrt{\frac{2}{\pi}}\, 2^n n^{-\frac{1}{2}}(1+O(n^{-1})) < \log_2 f(n) < \sqrt{\frac{2}{\pi}}\, 2^n n^{-\frac{1}{2}} \log_2 \sqrt{\frac{n\pi}{2}}\,(1+O(n^{-1}))$$

(Theorem 2), which in particular implies that for an arbitrary positive constant δ

$$2^n n^{-\frac{1}{2}-\delta} < \log_2 f(n) < 2^n n^{-\frac{1}{2}+\delta}$$

if n is sufficiently large, and that