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On the uniform continuity of Wiener process

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It is the purpose of this note to ameliorate Lévy's result concerning the uniform continuity of Wiener process. Let $\varphi(h)$ be a continuous and monotone increasing function which tends to zero with h. After P. Lévy we say that a function $f(t)$ verifies "*Hölder's weak condition*" relative to $\varphi(t)$, if there exists a positive number ε such that $|t'-t| = h \leqq \varepsilon$ yields the relation

$$|f(t')-f(t)| \leqq \varphi(h).$$

Let us put

$$\varphi_c(h) = \{h(2\log 1/h + c\log\log 1/h)\}^{1/2}.$$

Then we obtain the following theorem.

THEOREM. *If $c>5$, Wiener process $\{X(t,\omega);\ 0\leqq t\leqq 1\}$*[1] *verifies "Hölder's weak condition" relative to $\varphi_c(t)$ and if $c<-1$, it does not verify the condition, with probability one.*

PROOF. Let us put

$$\alpha(h) = Pr\{|\triangle X(t)| > \varphi_c(h)\}, \tag{1}$$

where $\triangle X(t)$ is the difference of $X(t+h)$ and $X(t)$. Since $\triangle X(t)$ is a normal random variable satisfying the conditions $E(\triangle X(t))=0$ and $V(\triangle X(t))=h$[2] we have the following asymptotic relation

$$\alpha(h)/h \sim (1/\pi)^{1/2}(\log 1/h)^{-(c+1)/2}. \tag{2}$$

If $c<-1$, we obtain

$$\alpha(h)/h \to \infty \quad \text{as} \quad h\to 0, \tag{3}$$

and therefore

$$1-(1-\alpha(1/n))^n \to 1 \quad \text{as} \quad n\to\infty. \tag{4}$$

1) ω is the probability parameter.

2) E and V denote the expectation and the variance respectively.