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On algebraic group varieties.

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Let G be a linear algebraic group of dimension r. We propose to study the brational type of G as an algebraic variety, i. e. the structure of the field $\mathfrak{R}(G)$ of rational functions on G. In his " Traité d'analyse " (vol. 3, p. 550), E. Picard shows that it is possible to express the coordinates of a point of G as rational functions of r parameters. The proof depends however on the basic field of the group being that of complex numbers; moreover, it is not proved that the parametric representation one obtains is proper (i. e. that a given point of the group corresponds in general to only one system of values of the parameters). In other words, the proof shows that $\mathfrak{R}(G)$ is contained in a purely transcendental extension of transcendence degree r of the basic field, but not that it is itself purely transcendental.

In this paper, we shall prove that, for any basic field (of characteristic 0), the field $\mathfrak{R}(G)$ is contained in some purely transcendental extension of transcendence degree r of the basic field. We shall also establish that, when the basic field is algebraically closed, $\mathfrak{R}(G)$ is itself purely transcendental, while, if the basic field is arbitrary, it may happen that $\mathfrak{R}(G)$ is not purely transcendental.

In what follows, we denote by G an irreducible algebraic group of dimension r composed of automorphisms of a finite dimensional vector space V over a field K of characteristic 0; we denote by $\mathfrak{g}$ the Lie algebra of G.

I. A reduction of the problem.

Let $\mathfrak{n}$ be the largest ideal of $\mathfrak{g}$ composed of nilpotent endomorphisms of V ([1], V, th. 3, 2). Then we have a direct sum decomposition $\mathfrak{g}=\mathfrak{n}+\mathfrak{b}+\mathfrak{s}$ of $\mathfrak{g}$ with the following properties: $\mathfrak{b}$ is an algebraic abelian subalgebra of the radical of $\mathfrak{g}$ whose elements are semi-simple; $\mathfrak{s}$ is a semi-simple subalgebra of $\mathfrak{g}$ whose elements commute with those of $\mathfrak{b}$