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On the multiplicative group of simple algebras.

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Let A be a central simple algebra of finite dimension over a commutative field F which contains an infinite number of elements. Let B be a subalgebra of A different from both A and F. A subalgebra B' is called *conjugate* to B if there exists a regular element t of A such that $B'=tBt^{-1}$. If we denote by $[B]$ the totality of subalgebras of A conjugate to B, the multiplicative group A^* of regular elements of A may be regarded as a transitive group of substitutions on $[B]$ in a natural manner, and every element of the subgroup F^* of A^* (the multiplicative group of regular elements of F) gives rise to the identity substitution. Now, we have

THEOREM. *F^* is precisely the kernel of the representation of A^* as a group of substitutions on $[B]$.*

This was proved previously by one of the writers in case where B is a simple subalgebra of A, and was applied to the structure-problem of the three dimensional rotation groups [3]. Our aim in the present paper is to show that the theorem is valid in the general form as above, and can be proved in even simpler way than in [3].

§1. We need a simple lemma on Kronecker product.

LEMMA. *Let B and C $(\neq F)$ be algebras with identity over F, and $A=B\times C$ their Kronecker product over F. If $t=b+c(b\in B,\ c\in C,\ c\notin F)$ is a regular element of A, we have $B\cap tBt^{-1}=V_B(b)$, where $V_B(b)$ denotes the set of all elements of B commutable with b.*

PROOF. If $x\in tBt^{-1}$, there exists $y\in B$ such that $(b+c)y=x(b+c)$, or equivalently, $(by-xb)\cdot 1=(x-y)c$. If, further, $x\in B$, we have $by=xb$ as well as $x=y$ in virtue of the linear disjointness of B and C over F. Hence $x\in V_B(b)$, i.e. $B\cap tBt^{-1}\subseteq V_B(b)$. Conversely, it is easily verified that $V_B(b)\subseteq B\cap tBt^{-1}$.

Now we proceed to the proof of the theorem. Let $N(B)$ be the totality of those regular elements of A which give rise to the identity