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On Weierstrass-Stone's theorem.

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Let Ω be a compact Hausdorff space and $C(\Omega)$ the ring of all real-valued continuous functions on Ω.

We define the norm for an element f of $C(\Omega)$ as

$$\|f\|=\sup_{x\in\Omega}|f(x)|,$$

then we have a Banach algebra $C(\Omega)$.

Weierstrass-Stone's theorem may be formulated as follows:

Let B be a subring in $C(\Omega)$ which has the following properties:

(1) *if $x_1, x_2, x_1 \neq x_2$ are arbitrary elements of Ω, then we can find an element f in B such that $f(x_1)\neq f(x_2)$.*

(2) *B has the unit* 1.

Then B is norm-dense in $C(\Omega)$.

In this theorem, a point of Ω may be considered as a linear functional on $C(\Omega)$. So, we shall consider here generally by what kind of systems of linear functionals on $C(\Omega)$ the set Ω in (1) can be replaced.

DEFINITION. Let $C^*(\Omega)$ be the set of all linear functionals on $C(\Omega)$. A subsystem $\mathfrak{S}$ of $C^*(\Omega)$ is said to satisfy the condition of Weierstrass (shortly W-condition) if $\mathfrak{S}$ satisfies the following condition:

If B is an arbitrary subring of $C(\Omega)$ which contains the unit 1, and if for any two different elements φ, ψ of $\mathfrak{S}$ there exists an f in B such that $\varphi(f)\neq\psi(f)$, then B is norm-dense in $C(\Omega)$.

Clearly the totality of point-functionals satisfies this condition.

LEMMA 1. *Let F be a linear space and φ, ψ two linear functionals on F. If $\varphi(f)=0$ always implies $\psi(f)=0$, then we can find a real number a such that $\psi(f)=a\,\varphi(f)$.*

PROOF. Let f_0 be an element of F such that $\varphi(f_0)\neq 0$. If we can not find such an element, this lemma follows trivially.

Since $\varphi\{\varphi(f_0)f-\varphi(f)f_0\}=0$ $(f\in F)$, we have by assumption,

$$\psi\{\varphi(f_0)f-\varphi(f)f_0\}=0 \qquad (f\in F),$$

i.e.

$$\varphi(f_0)\,\psi(f)-\varphi(f)\,\psi(f_0)=0 \qquad (f\in F).$$