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A note on Kummer extensions

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1. Let k be an arbitrary field and $Z(k)$ the set of all integers $n \geqq 1$ such that k contains a primitive n-th root of unity. It is clear that, if $Z(k)$ contains m and n, it also contains the least common multiple of these two integers. Therefore the set of all rational numbers with denominators in $Z(k)$ is an additive group $R(k)$ containing the group of all integers Z, and the quotient group $\bar{R}(k)=R(k)/Z$ is isomorphic with the multiplicative group $W(k)$ of all roots of unity in k.

We now take an algebraic closure Ω of k and consider the subfield K of Ω obtained by adjoining all $\alpha^{1/n}$ to k, where α is an arbitrary element in k and n is an arbitrary integer in $Z(k)$. K is obviously the composite of all finite Kummer extensions of k contained in Ω and hence, may be called the *Kummer closure* of k in Ω. K/k is clearly an abelian extension and its structure is independent of the choice of the algebraic closure Ω of k. In particular, the structure of the Galois group $G(K/k)$ of K/k is an invariant of the field k, and we shall show in the following how we can describe it by means of groups which depend solely on the ground field k.

2. We shall first define a symbol (σ, α, r) for arbitrary σ in $G=G(K/k)$, $\alpha \neq 0$ in k and r in $R(k)$. Namely, we express r as a fraction $\frac{m}{n}$ with denominator n in $Z(k)$ and choose an elemnt a in K such that $a^n=\alpha^m$. The symbol (σ, α, r) is then defined by

$$(\sigma, \alpha, r)=a^{\sigma-1}.$$

It is easy to see that (σ, α, r) is an n-th root of unity in k and is independent of the choice of the fractional expression $\frac{m}{n}$ of r and, also, of the choice of a in K such that $a^n=\alpha^m$.