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Integration of the equation of evolution in a Banach space.

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§ 1. Introduction. Theorems.

In what follows we consider the integration of the equation of evolution[1]

(1.1) $$dx(t)/dt=A(t)x(t)+f(t), \qquad a\leqq t\leqq b,$$

and the associated homogeneous equation

(1.1′) $$dx(t)/dt=A(t)x(t).$$

Here the unknown $x(t)$ is an element of a complex Banach space $\mathfrak{B}$ depending on a real variable t, while $f(t)$ is a given element of $\mathfrak{B}$ and $A(t)$ is a given, in general unbounded, linear operator in $\mathfrak{B}$, both depending on t

The solution of (1.1) is formally given by

(1.2) $$x(t)=U(t,a)x+\int_a^t U(t,s)f(s)\,ds, \qquad x=x(a),$$

where $U(t,s)$ is a linear operator in $\mathfrak{B}$ depending on s,t with $s\leqq t$. The main purpose of the present paper is to give some sufficient conditions for the existence of $U(t,s)$ and to study its properties.

If $A(t)=A$ is independent of t, $U(t,s)$ is given formally by $U(t,s)=\exp[(t-s)A]$, and the rigorous definition of the exponential function has been given by Hille and Yosida in connection with the analytical theory of semi-groups.[2] As we are going to generalize some of their results to the case in which $A(t)$ actually depends on t, it is natural to take over their assumptions on the infinitesimal generator A for our

1) The terminology after Schwarz [2].

2) Hille [1], Chap. XII, in particular Theorem 12.2.1; Yosida [3], Theorem 2.