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Note on Betti numbers of Riemannian manifolds I.

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In this paper, we give some applications of a theorem of Bochner—Lichnerowicz on the Betti numbers of a Riemannian manifold. We consider a Riemannian manifold R_n whose fundamental tensor g_{ij} is positive definite and assume that R_n is compact and orientable.

THEOREM I. (BOCHNER-LICHNEROWICZ)

In R_n, if the quadratic form

$$\left(\frac{p-1}{2}R_{ijkl}+R_{ik}g_{jl}\right)f^{ij}f_{kl} \quad (f^{ij}=-f^{ji}) \tag{1}$$

is everywhere positive semi-definite, then, for any harmonic tensor $X_{i(1)\cdots i(p)}$ of degree p, it holds that

$$X_{i(1)\cdots i(p);r}=0\,,$$

and hence we have

$$B_p \leqq \binom{n}{p}$$

where B_p denotes the p-th Betti number and $p \geqq 2$.

When $p=1$, the quadratic form (1) *can be replaced by*

$$R_{ij}f^i f^j\,, \tag{2}$$

and if this form is everywhere positive semi-definite, then the covariant derivative of any harmonic vector vanishes, and hence we have

$$B_1 \leqq n\,.$$

If the quadratic form (1) *or* (2) *is everywhere positive definite, then the harmonic vector or tensor should be identically zero, and hence we have*

$$B_p=0 \text{ or } B_1=0\,.$$