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On the uniform distribution of numbers mod. 1.

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1. Let $\{x_n\}$ $(n=1, 2, \cdots)$ be a sequence of real numbers and put

$$\bar{x}_n = x_n - [x_n], \quad 0 \leqq \bar{x}_n < 1. \tag{1}$$

Let I be an interval in $[0, 1]$ and $|I|$ be its length and $n(I)$ be the number of $\bar{x}_\nu$ $(\nu=1, 2, \cdots, n)$ contained in I. If for any I

$$\lim_{n\to\infty} \frac{n(I)}{n} = |I|, \tag{2}$$

then $\{x_n\}$ is called to be uniformly distributed mod. 1.

The following theorems are known.

THEOREM 1 (Weyl)[1]. *The necessary and sufficient condition that $\{x_n\}$ is uniformly distributed mod. 1 is that for any R-integrable function $f(x)$ in $[0, 1]$,*

$$\lim_{n\to\infty} \frac{f(\bar{x}_1) + \cdots + f(\bar{x}_n)}{n} = \int_0^1 f(x)dx.$$

THEOREM 2 (Weyl)[2]. *The necessary and sufficient condition that $\{x_n\}$ is uniformly distributed mod. 1 is that for $m=0, \pm 1, \pm 2, \cdots$*

$$\sum_{\nu=1}^{n} e^{2\pi m x_\nu i} = o(n).$$

THEOREM 3 (van der Corput)[3]. *Let $g_h(t) = g(t+h) - g(t)$ $(h=1, 2, \cdots)$. If $\{g_h(n)\}$ is uniformly distributed mod. 1 for any h, then $\{g(n)\}$ is uniformly distributed mod. 1.*

1), 2). H. Weyl: Über die Gleichverteilung von Zahlen mod. 1, Math. Ann. 77 (1916).

3) J. G. van der Corput: Diophantische Ungleichungen, I, Zur Gleichverteilung modulo Eins, Acta Math. 56 (1931).