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Betti numbers and exact differential forms.

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In this paper we consider an orientable compact positive definite Riemannian space R_n. The relations between exact differential forms and Betti numbers are known as de Rham's theorem. We shall give some applications of this theorem.

In § 1 and § 2, we shall find the conditions that the Betti number be greater than a certain number. In § 3, we consider the conditions that the equations of harmonic tensors become some total differential equations which enable us to evaluate the Betti numbers.

§ 1. LEMMA 1. *Let $A_{i(1)\cdots i(p)}$ and $C^{i(1)\cdots i(p)}$ be skew-symmetric and satisfy the conditions*

$$(1.1) \qquad A_{i(1)\cdots i(p)}=B_{[i(1)\cdots i(p-1);\, i(p)]}$$

and

$$(1.2) \qquad C^{i(1)\cdots i(p)}{}_{;\, i(p)}=0\,,$$

where $B^{i(1)\cdots i(p-1)}$ is a certain tensor. Then it follows that

$$(1.3) \qquad \int A_{i(1)\cdots i(p)}C^{i(1)\cdots i(p)}dv=0\,,$$

where dv is the volume element and the integral extends over the whole space.

PROOF. By Green's theorem we have

$$(1.4) \qquad 0=\int (B_{i(1)\cdots i(p-1)}C^{i(1)\cdots i(p)})_{;\, i(p)}dv=\int B_{i(1)\cdots i(p-1);\, i(p)}C^{i(1)\cdots i(p)}dv$$
$$+\int B_{i(1)\cdots i(p-1)}C^{i(1)\cdots i(p)}{}_{;\, i(p)}dv=\int B_{[i(1)\cdots i(p-1);\, i(p)]}C^{i(1)\cdots i(p)}\, dv$$
$$=\int A_{i(1)\cdots i(p)}C^{i(1)\cdots i(p)}dv\,.$$

THEOREM 1. *Let $H_{(A)i(1)\cdots i(p)}$ $(A=1,\ 2,\ \cdots,\ s)$ and $H_{(B)i(p+1)\cdots i(n)}$ $(B=1,\ 2,\ \cdots,\ t)$ be exact and put*