Journal of the Mathematical Society of Japan Vol. 4, No. 1, July, 1952.

Fundamental theorems in potential theory.

By Masatsugu Tsuji

(Received April 5, 1952)

The potential theory plays an important rôle in function-theory, so that in this paper, I shall prove fundamental theorems in potential theory in the shortest lines. Almost all results are known and the proofs are not new, but are somewhat simpler than the usual ones.

Theorem 20 seems to be new and of some interest. In view of applications to function-theory, we confine ourselves to logarithmic potentials.

1. Maximum principle.

Let F be a bounded closed set on the z-plane and $\mu(e) \geqq 0$ be a positive mass distribution on F of finite total mass and consider the potential:

$$u(z)=\int_F \log \frac{1}{|z-a|}\, d\mu(a)\,.$$

THEOREM 1. (*Maximum principle*).[1] *If $u(z) \leqq K$ on F, then $u(z) \leqq K$ in the whole z-plane.*

PROOF. Let D be the complement of F and $a_0 \in F$ be its boundary point. It is sufficient to prove

$$\overline{\lim_{z \to a_0}}\, u(z) \leqq K \qquad (z \in D)\,.$$

Let D_ρ be the part of D contained in $|z-a_0|<\rho$ and F_ρ be that of F

1) For Newtonian potentials: M. A. Maria: The potential of a positive mass and the weight function of Wiener. Proc. Nat. Acad. Sci. U S. A. 20 (1934). For general potentials: O. Frostman: Potentiel d'équilibre et capacité des ensembles, Lund (1935). Frostman's proof depends on Poincaré's sweeping-out process. A simple proof independent of the sweeping-out process was given by Y. Yosida: Sur le principe du maximum dans la théorie du potentiel. Proc. Imp. Acad. 17 (1941).