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On the Theory of Radicals in a Ring

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Introduction. In generalizing the classical notion of the radical in a ring, different kinds of radicals have been defined by many authors, including Azumaya [1],[0] Baer [2], Brown-McCoy [4], Jacobson [6], Köthe [9], Levitzki [10] and McCoy [12]. The main purpose of the present paper is to give a unified treatment to these theories, though we do not claim that we cover all of them, to introduce some new kinds of radicals, and further to study the properties of these radicals, both new and old. Our main concern is about p-radical, J-radical, e-radical and M-radical, each being a special case of C-radical; here p-radical and J-radical are the radicals in the sense of McCoy [12] and Jacobson [6] respectively.

In § 1, we first introduce a general concept of C-radical, and the notation $(1,R)$ which is a typical over-ring of a ring R containing identity. Further, we prove a fundamental characterization of prime ideals, and we define, by the way, the concept of irreducible ideals. In § 2, we introduce the notions of m-systems and m-families. A simmilar concept of m-systems has been already defined by McCoy [12], and ours is its modification (cf. foot-note 1); it plays, combined with the concept of prime ideals, an important rôle in this paper. In § 3, we treat semi-prime ideals, and define the concept of p-radical. Further, a fundamental characterization of semi-primeness (Proposition 8) is proved and we see that a radical ideal in the sense of Baer [2] is a semi-prime nil-ideal (and conversely) and therefore his lower radical coincides with our p-radical. § 4, is mostly devoted to J-radical, considered as a special case of C-radical, while in § 5, we introduce a concept of e-radical, and in § 6, we study some properties of J-radical and e-radical concerning idempotent elements. In § 7, we define M-radical and M-quasi-radical; the latter does not coincide with the classical notion of radical even when minimum condition is assumed for right ideals; for this reason we use the term quasi-radical. In § 8, we study the correspondences between radicals of R and those of $(1,R)$. In §9, we observe another typical over-ring $[1,R]$ of a ring $[R]$ which contains the identity. Further, we study

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(0) The numbers in brackets refer to the bibliography at the end.