Journal of the Mathematical Society of Japan Vol. 3, No. 2, Dec. 1951.

Notes on Fourier Analysis (XXIX): An Extrapolation Theorem

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(Received Nov. 21, 1949)

1. Introduction. Let $f(x)$ be a real measurable function defined in the interval $(0,2\pi)$ we write $f(x) \in L^p (p>0)$ when $|f(x)|^p$ is integrable in $(0,2\pi)$, and $f(x) \in L^{*k}(a>0)$ when $|f(x)|\log^k(1+f^2(x))$ is integrable in $(0,2\pi)$. L^{*1} is the function class which was introduced by A. Zygmund [1].

In the theory of Fourier series, the transformations of the following type play an important rôle: that is, $T[f(x)]=g(x)$ transforms every integrable function $f(x)$ to another $g(x)$, both being defined in $(0,2\pi)$, such that the inequalities

$$\{\int_0^{2\pi} T[f(x)]|^p\, dx\}^{1/p} \leqq A_p\{\int_0^{2\pi}|f(x)|^p\, dx\}^{2/p} \qquad (p>1), \tag{1.1}$$

and

$$\int_0^{2\pi}|T[f(x)]|\, dx \leqq A_k| \int_0^{2\pi} f(x)|\ \log^k(1+f^2(x))\, dx + B_k \tag{1.2}$$

hold where A_p, A_k, B_k are constants depending only on p, k and k, respectively.

The inequalities (1.1) and (1.2) are usually proved independently. We shall now give a general principle to deduce the inequality of the type (12) from that of the type (1.1). That is,

Theorem. *Let T be a transformation which transforms every integrable function to a measurable function, both being defined in a finite interval (a,b), such that* (i)

$$f(x)=\sum_{\nu=0}^{\infty} f_\nu(x) \text{ implies } |T[f]| \leqq \sum_{\nu=0}^{\infty}|T[f_\nu]| \tag{1.3}$$

and

$$|T[f]=|T[-f]|, \tag{1.4}$$

(ii) *the inequality*

$$\{\int_a^b |T[f]|^p\, dx\}^{1/p} \leqq A_p\{\int_a^b |f(x)|^p\, dx\}^{1/p} \tag{1.5}$$

holds with the constant A_p satifying the inequality

$$A_p \leqq A/(p-1)^k \tag{1.6}$$

for all p, $1<p\leqq 2$, for some $k>0$, and for a constant A depending only on