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On the Measure-Preserving Flow on the Torus[1)]

Toshiya SAITO

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1. Let us consider the one-parameter stationary flow S_t on the euclidean plane defined by the following system of differential equations

$$
(1) \qquad \begin{cases} \dfrac{dx}{dt} = X(x, y), \\[2mm] \dfrac{dy}{dt} = Y(x, y), \end{cases}
$$

where X and Y are assumed to be real-valued functions having continuous first derivatives. If we moreover assume X and Y to be periodic functions of period 1 with respect to their arguments, they can be expanded into uniformly convergent Fourier series in the following way.

$$
(2) \qquad \begin{cases} X = \sum a_{mn} e^{2\pi i (mx+ny)} \\ Y = \sum b_{mn} e^{2\pi i (mx+ny)}. \end{cases}
$$

Let us then suppose that our flow is measure-preserving, or, in other words, differential equations (1) admit an integral invariant

$$
\iint dx \; dy.
$$

In this case, we have

$$
(3) \qquad \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 0.
$$

Then, by termwise differentiation, this relation can be written in the form

$$
m a_{mn} + n b_{mn} = o, \quad m, n = o, \pm 1, \pm 2, \cdots.
$$

Hence we can find a sequence $\{c_{mn}\}$ such that

$$
a_{mn} = n c_{mn}, \quad b_{mn} = -m c_{mn}, \quad (m, n) \neq (o, o).
$$

Consequently we can write

$$
(2') \qquad \begin{cases} X = a_{00} + \sum n c_{mn} e^{2\pi i (mx+ny)}, \\ Y = b_{00} - \sum m c_{mn} e^{2\pi i (mx+ny)}. \end{cases}
$$