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On the Algebraic Structure of Group Rings

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§ 1. Introduction

1. Let $\mathfrak{G}$ be a group of finite order g. If K is any given field of characteristic 0, the group ring Γ of $\mathfrak{G}$ with regard to K is a semisimple algebra. By Wedderburn's theorems, Γ is a direct sum of simple algebras A_i;

$$\Gamma=\mathsf{A}_1\oplus\mathsf{A}_2\oplus\cdots\oplus\mathsf{A}_s. \tag{1}$$

Each A_i is isomorphic to a complete matric algebra of a certain degree q_i over a division algebra Δ_i;

$$\mathsf{A}_i\cong[\Delta_i]_{q_i}. \tag{2}$$

The center Z_i of A_i may also be considered as the center of Δ_i. It is an extension field of finite degree r_i over K. Since Δ_i then is a central simple algebra over Z_i, its rank over Z_i is the square of a natural integer m_i. Then A_i has the rank $r_i q_i^2 m_i^2$ over K. We shall call the numbers m_i the *Schur indices* of $\mathfrak{G}$, since they first occurred in the work of *I. Schur* on representations of $\mathfrak{G}$ by linear transformations.

2. The theory of representations of groups of finite order was developed originally by *Frobenius* for the case that the coefficients of the representing linear transformations belong to an algebraically closed field of characteristic 0. The case of an arbitrary field K of characteristic 0 was considered by *I. Schur*.[1] We quote the main results.

Every representation of $\mathfrak{G}$ is completely reducible. Two representations of $\mathfrak{G}$ are similar, if and only if they have the same character. It is then sufficient to consider the irreducible representations of $\mathfrak{G}$ in K and their characters. These irreducible representations $\mathfrak{T}_1, \mathfrak{T}_2, \cdots, \mathfrak{T}_s$ are in one-to-one correspondence to the simple algebras $\mathsf{A}_1, \mathsf{A}_2, \cdots, \mathsf{A}_s$ in (1).

If $\bar{\mathsf{K}}$ is the algebraic closure of K, then $\mathfrak{T}_i$ breaks up in $\bar{\mathsf{K}}$ into r_i

1) Schur [1], [2]. The connections with the theory of algebras are given in Brauer [1], [2]. See also Albert [1]; van der Waerden [1], Chapter XVII, [2]; Weyl [1], Chapters III and X.