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A Note on Finite Ring Extensions

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Let $R \subset S$ be two commutative rings. We shall say that S is a modul finite extension of R if a finite number of elements $\omega_1, \omega_2, \cdots \omega_n$ of S can be found such that

$$S = R\omega_1 + R\omega_2 + \cdots + R\omega_n.$$

This modul finite extension has to be distinguished from what we shall call a ring finite extension

$$S = R[\xi_1, \xi_2, \cdots \xi_n],$$

in which every element of S can be written as polynomial in the generators $\xi_1, \xi_2, \cdots \xi_n$ with coefficients in R. If we call S' the ring of all polynomials in the indeterminates $x_1, x_2, \cdots x_n$ with coefficients in R then S is a homomorphic image of S' and the following well known lemma is immediate:

Lemma 1. If R is a Noetherian ring[1] with unit element and $S = R[\xi_1, \xi_2, \cdots \xi_n]$ a ring finite extension of R then S is Noetherian.

Lemma 2. Let R be a Noetherian ring with unit element and $S = R\omega_1 + R\omega_2 + \cdots + R\omega_n$ a modul finite extension of R. Then any intermediate ring T: $R \subset T \subset S$ is also a modul finite extension of R.

The proof is simple and also well known. We consider S as an R-space. The R-subspaces of S—and T is one of them—satisfy the ascending chain condition. T is therefore a modul finite extension of R.

The main result of our note is:

Theorem 1. Let R be a Noetherian ring with unit element, $S = R[\xi_1, \xi_2, \cdots \xi_n]$ a ring finite extension and T an intermediate ring such that S is a modul finite extension of T: $S = T\omega_1 + T\omega_2 + \cdots + T\omega_m$. Then T is a ring finite extension of R.

Proof: There exist expressions of the form:

$$(1) \qquad \xi_i = \sum_{\nu=1}^{m} a_{i\nu}\omega_\nu; \quad i = 1, 2, \cdots n; \qquad a_{i\nu} \in T$$

1) i.e. a ring with ascending chain condition for ideals.