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Integration of Fokker-Planck's Equation with a Boundary Condition

Kôsaku YOSIDA

1. Introduction. We consider Fokker-Planck's equation[1)]

$$(1)\qquad \frac{\partial f(t,x)}{\partial t}=Af(t,x),\ t\geqq 0,$$

$$(Af)(x)=\frac{1}{\sqrt{g(x)}}\frac{\partial^2}{\partial x^i\partial x^j}(\sqrt{g(x)}\,b^{ij}(x)f(x))$$
$$+\frac{1}{\sqrt{g(x)}}\frac{\partial}{\partial x^i}(-\sqrt{g(x)}a^i(x)f(x))$$

in a connected region R of an n-dimensional orientable Riemannian space with the metric $ds^2=g_{ij}(x)dx^idx^j$. As usual, the volume element in R is defined by $dx=\sqrt{g(x)}dx^1dx^2\cdots dx^n, g(x)=det(g_{ij}(x))$. We assume that the contravariant tensor $b^{ij}(x)$ be such that $b^{ij}(x)\xi_i\xi_j>0$ in R (for $\sum_i \xi_i^2>0$). The $a^i(x)$ obeys, by the coordinate change $x\rightarrow\bar{x}$, the transformation rule

$$(2)\qquad \bar{a}^i(\bar{x})=\frac{\partial\bar{x}^i}{\partial x^k}a^k(x)+\frac{\partial^2\bar{x}^i}{\partial x^k x^s}b^{ks}(x).$$

These propeities of the coefficients $a^i(x)$ and $b^{ij}(x)$ are connected with the probabilistic meaning of the equation (1).

We assume that $g_{ij}(x)$, $a^i(x)$ and $b^{ij}(x)$ are infinitely differentiable functions of the coordinates $x=(x^1, x^2, \cdots, x^n)$. The purpose of the present note is to consider a certain natural boundaiy condition on the boundary ∂R of R for the probability density $f(t, x)$ at the time moment $t>0$ and to discuss, for this boundary condition, the stochastic integrability (in the sense to be explained in § 3) of the equation (1). As in the previous papers, our treatment and the method of proof relies upon the theory of semi-group of linear operators,[2)] which is, so to speak, an operator-theo-

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1) A. Kolmogoroff: Zur Theorie der stetigen zufälligen Prozess, Math. Ann., **108** (1933), 149–160. K. Yosida: An extension of Fokker-Planck's equation, Proc. Japan Acad., **25** (1949), (9), 1–3.

2) E. Hille: Functional Analysis and Semi-groups, New York (1948). K. Yosida: On the differentiability and the representation of one-parameter semi-group of linear operators, Journ. Math. Soc. Japan, **1** (1949), 1, 15–21, and K. Yosida: An operator-theoretical treatment of temporally homogeneous Markoff process, ibid., **1** (1949), 1, 244–235.