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Some Remarks on Relatively Free Homotopy.

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Consider an arcwise connected topological space Z and select one of its points $*$ as a base point. Suppose furthermore that there is given an arcwise connected subspace Y of Z containing the base point $*$. Given a point $*'$ of Y, which may or may not be distinct from $*$, a path component of Y, i. e. a homotopy class of paths from $*$ to $*'$, induces an isomorphism between two n-th relative homotopy groups $\pi_n(Z, Y, *)$ and $\pi_n(Z, Y, *')$, attached to two points $*$, $*'$ respectively. If in particular $* = *'$, every element of the fundamental group $\pi_1(Y, *)$ induces an automorphism of the group $\pi_n(Z, Y, *)$, and therefore, algebraically speaking, the former may be regarded as a group of operators on the latter. Now I shall define a homotopy group $\sigma_n(Z, Y, *)$ for every integer $n \geqq 3$, containing subgroups isomorphic to $\pi_n(Z, Y, *)$ and $\pi_1(Y, *)$, in which the operation of $\pi_1(Y, *)$ on $\pi_n(Z, Y, *)$ forms an inner automorphism. As is seen later, an element of the group σ_n can be represented by a continuous mapping belonging to Z^{E^n} which transforms $S^{n-1}=\dot{E}^n$ into Y and to different points on S^{n-1} into the base point $*$. (E^n means an n-dimensional cube, see foot note) The pair (Z, Y) is usually called "relatively n-simple," if $\alpha^\xi=\alpha$ for any element ξ of $\pi_1(Y, *)$ and any α belonging to $\pi_n(Z, Y, *)$, and it is well known that in such a pair of spaces a base point $*$ can be arbitrarily selected in Y, in the sense that the isomorphism between two groups $\pi_n(Z, Y, *)$ and $\pi_n(Z, Y, *')$ attached to an aibitraiily chosen point $*'$ in Y is determined indepently of the path connecting $*$ to $*'$. Therefore the simplicity of a pair of spaces may be considered as an intrinsic property of the pair. A pair (Z, Y) which is relatively n-simple is characterized by the purely algebraic relation in σ_n : $\sigma_n(Z, Y, *)$ is isomorphic to the direct product of two groups $\pi_n(Z, Y, *)$ and $\pi_1(Y, *)$. This paper will contain these and some other remarks obtained by applying M. Abe's arguments in (1) to the case of relative homotopy groups.

1. *Definition of* $\sigma_n(Z, Y, *)$ *for* $n \geqq 3$.

Let $e(x_0)$, $1 \geqq x_0 \geqq 0$, be a $*$-based loop in Y. Denote by σ_n the

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1) $E^n = x^n\ (x_0, x_1, \cdots\cdots, x_{n-1})$; $1 \geqq x_1 \geqq 0$, $n-1 \geqq i \geqq 0$,
$x^n{}_i = (x_i\ , x_{i+1} \cdots\cdots x_{n-1})$