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On Maximal Proper Sublattices.

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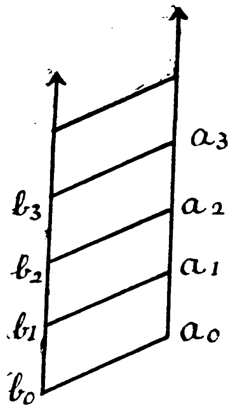

Fig. 1.

G. Birkhoff has proposed the following problem in his revised edition of "Lattice Theory".

Problem 18: Prove or disprove that every proper sublattice S of a lattice L can be extended to a maximal proper sublattice. He suggests: The answer may be yes for distributive lattices.

In this paper we shall prove that the answer is yes for any Boolean algebra (with I and O). But this will be disproved for the distributive lattice $\{a_n, b_n\ (n=0,1,2,\cdots\cdots)\}$ with the Hasse diagram as Fig. 1. Consider, in fact, the sublattice $S=\{a_n(n=0,1,2,\cdots\cdots)\}$. Since the sublattice generated by S and b_n contains all $b_m(m \geqq n)$, S cannot be extended to a maximal proper sublattice.

Let L be a lattice, S a proper sublattice of L and x an element of $L-S$. $M_x(S)$ denotes a maximal subset among all the subsets of L containing S, such that the sublattices generated by them do not contain x. We shall write M_x for any $M_x(\phi)$, where ϕ is the empty set. The existence of $M_x(S)$ is assured by Zorn's lemma and it is evidently one of M_x.

Lemma: A maximal proper sublattice of L is characterized as a maximal subset of L among all the subsets M satisfying the following condition.

(*) There exists an element of L which is not contained in the sublattice generated by M.

Proof: A maximal M is a proper sublattice. Since every proper sublattice satisfies the condition (*), a maximal M is a maximal proper sublattice.

A maximal proper sublattice N satisfies the condition (*). Since every sublattice generated by a subset satisfying the condition (*) is a proper sublattice, N is a maximal M. Q. E. D.

Corollary: A maximal proper sublattice is characterized as a maximal element of the set of all M_x, $x \in L$.

Theorem: Let L be a Boolean algebra with I and O. Then every proper sublattice S of L can be extended to a maximal proper sublattice.