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On Baire's Theorem concerning a Function f(x, y), which is Continuous with respect to Each Variable x and y.

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The purpose of this paper is to give a simple proof of the following Baire's theorem[1].

Theorem. *Let $f(x, y)$ be defined in a square Δ: $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$ and be continuous with respect to each variable x and y. Then there exists a set X on the x-axis, which is dense on $[0,1]$, such that for any $x_0 \in X$, $f(x, y)$, considered as a function of two varibles (x, y), is continuous at every point of the segment $x=x_0$, $0 \leqq y \leqq 1$. Similarly there exists a set Y on the y-axis, which is dense on $[0, 1]$, such that for any $y_0 \in Y$, $f(x, y)$ is continuous on the segment $y=y_0$, $0 \leqq x \leqq 1$*

Proof. We will prove the existence of the set X, which satisfies the conditions of the theorem. The existence of the set Y can be proved similarly.

We define $f_n(x, y)$ $(n=1, 2\cdots)$ in Δ as follows:

$$f_n(x,y)=f(x,\frac{\nu}{2^n})+\frac{f(x,\frac{\nu+1}{2^n})-f\left(x,\frac{\nu}{2^n}\right)}{\frac{1}{2^n}}\left(y-\frac{\nu}{2^n}\right) \tag{1}$$

for $0 \leqq x \leqq 1, \frac{\nu}{2^n} \leqq y \leqq \frac{\nu+1}{2^n}$, $(\nu=0,1,2,\cdots\cdots,2^n-1)$.

Then $f_n(x, y)$ is continuous in Δ and

$$f(x, y)=\lim_{n\to\infty} f_n(x, y), \tag{2}$$

$$\lim_{n\to\infty} [\operatorname*{Max.}_{0\leqq y\leqq 1} |f(x, y)-f_n(x, y)|]=0, \text{ for a fixed } x. \tag{3}$$

From (2), it follows that $f(x, y)$ is of the first class of Baire.

For a fixed $\varepsilon>0$, we define a set $E_n(\varepsilon)$ on the x-axis by

$$E_n(\varepsilon)=\underset{x}{E}[\operatorname*{Max.}_{0\leqq y\leqq 1} |f(x, y)-f_n(x, y)| \leqq \varepsilon], \tag{4}$$

1) R. Baire: Sur les fonctions de variables réelles. Annali di Matematica. (3) 3 (1899). K. Bögel: Über die Stetigkeit und die Schwankung von Funktionen zweier Veränderlichen. Math Ann. 81 (1920).