Journal of the Mathematical Society of Japan Vol. 2, Nos. 1–2, Sept. 1950

On the mapping functions of Riemann surfaces.

Tokunosuke YOSIDA.

(Received Apr. 21, 1949)

Let W be a simply connected infinitely many sheeted open Riemann surface, whose singularities are all logarithmic and lie only on a finite number of base-points $x_1, x_2, \ldots, x_n$ $(n \geqq 3)$, and W^∞ be its universal covering surface.

Let $x = m(z)$ be the function which maps W^∞ one-to one and conformally on the unit-circle $|z|<1$. The properties of the function $m(z)$ are well known. Let $x=\varphi(w)$ be the function which maps W one-to-one and conformally on the finite plane $w \neq \infty$ or the unit-circle $|w|<1$ according as W is parabolic or hyperbolic and $\varphi^{-1}(x)$ be its inverse function. We shall obtain some properties of the function $\varphi^{-1}(m(z))$, which is regular in $|z|<1$.

Let

$$w=f(z)=\varphi^{-1}(m(z)) \tag{1}$$

and R be the Riemann surface on which the unit-circle $|z|<1$ is mapped one-to-one and conformally by $w=f(z)$. If W is of parabolic type, then R is a Riemann surface spread over the w-plane. If W is of hyperbolic type, then R is a Riemann surface spread over the unit-circle $|w|<1$. Let B be the boundary of the domain of $\varphi(w)$. The set B consits of only the point at infinity or the all points on the circumference $|w|=1$ according as the Riemann surface W is parabolic or hyperbolic.

Lemma 1. *The set M of points on the w-plane, which are the projections of the branch points of R is enumerable and the set M' of the limiting points of M is contained in the set B.*

Proof. Since the branch points of the universal covering surface W^∞ lie only on the base-points $x_1, x_2, \ldots, x_n$ and $f^{-1}(w)=m^{-1}(\varphi(w))$ by the relation (1), we have a regular functional element of $f^{-1}(w)$ at the point w, if $\varphi(w) \neq x_i (i=1,2,\ldots,n)$. Hence the projections of the branch points of R on the w-plane are the zero-points of $\varphi(w)-x_i$ $(i=1,2,\ldots,n)$. As the zero-points of an analytic function is enumerable, the set M is enumerable.

Since the limiting point of the zero-points of an analytic function lies on the boundary of the domain of definition, the set M' is contained in the set B.