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On a Generalization of Fubini's Theorem and Its Application to Green's Formula

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The object of the present paper is to establish a generalization of Fubini's theorem in the theory of integrals and to apply it to the proof of Green's formula under considerably general conditions.

The usual form of Fubini's theorem is concerned with the transformation of the integral of a summable function over the Euclidean space R_{p+q}(p and q natural numbers) into a repeated integral taken over R_p and R_q successively, the space R_{p+q} being the cartesian product of the spaces R_p and R_q. But this last circumstance is not essential for the validity of the theorem. In fact, we may take, roughly speaking, any two spaces Φ and Ψ with measures μ and ν respectively, define a mapping φ of Φ onto $\Psi=\varphi(\Phi)$, and denoting by Φ_y the inverse image $\varphi^{-1}(y)$ of $y \in \Psi$ under the mapping φ and by μ_y a measure on Φ_y, we have the formula (see Theorem 4)

$$\int_{\Phi} f(x)\, d\mu(x)=\int_{\Psi}\left[\int_{\Phi_y} f(x)\, d\mu_y(x)\right] d\nu(y)$$

for every $f(x)$ non-negative and measurable on Φ, provided that certain conditions involving the three measures μ, ν and μ_y are satisfied.

There is a research by P.R.Halmos [7] along similar lines of idea, but it seems to us that there is little point of contact between his paper and ours, since Halmos's interest lies chiefly in other directions.

Utilizing the generalized Fubini theorem thus established, we shall be able to prove our main theorem (Theorem 7) on the transformation of a Stieltjes integral into an ordinary one. In case the function $G(x)$ with respect to which we integrate is monotone, this is a well-known theorem and in fact is taken by Hobson (see Hobson [8], p. 605) as the very definition of the Stieltjes integral; but our theorem is concerned with a general function $G(x)$ of bounded variation and our result seems to be