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On the Multivalency of Analytic Functions

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The purpose of the present note is to extend NOSHIRO's theorem[1] (generalization of DEUDONNÉ's theorem[2]) concerning the univalency of analytic functions to the case of n-valence. First, assuming that $\varphi(z)=c_0+\dots\dots$, $c_0\neq 0$, is regular and $\varphi(z)\subset D$[3] in $|z|<1$, where D is a given connected domain, we obtain a general theorem on the multivalency and the star-shapedness of the function $f(z)\equiv z^n\varphi(z)$, according to K. NOSHIRO's method with the aid of KAKEYA's principle[4]. Then, we shall give some consequences of this theorem, taking some special domains as D, one of which gives a result obtained by Lynn H. LOOMIS[5].

Lemma. *Suppose that* $\varphi(z)=c_0+\dots\dots$, $c_0\neq 0$, *is regular in* $|z|<1$. *Then* $f(z)\equiv z^n\varphi(z)$ *is n-valent and starshaped with respect to the origin for* $|z|<1$, *provided that*

$$\Re\left(z\frac{f'(z)}{f(z)}\right)>0. \qquad (|z|<1). \tag{1}$$

Proof. If $\Re\left(z\dfrac{f'(z)}{f(z)}\right)>0$ $(|z|<1)$, then $f(z)$ does not vanish in the unit circle except at the origin and so there exists a function $h(z)$ which is regular in $|z|<1$ and satisfies

$$f(z)\equiv z^n\varphi(z)=[h(z)]^n.$$

Consequently

$$f'(z)=n[h(z)]^{n-1}h'(z), \qquad z\frac{f'(z)}{f(z)}=nz\frac{h'(z)}{h(z)}.$$

By (1) we have

$$\Re\left(z\frac{h'(z)}{h(z)}\right)>0 \qquad (|z|<1). \tag{2}$$

As is well known, (2) is a sufficient condition in order that a function