Journal of the Mathematical Society of Japan Vol. 2 Nos. 1-2, Sept., 1950.

On Betti Numbers of Riemannian Spaces.

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(Received June. 15, 1949)

1. Recently H. Iwamoto [1] has proved the following

Theorem. *Let B_P be the p-th Betti number of an orientable compact positive definite Riemannian space and let B'_P be the maximum number of linearly independent skew-symmetric tensors of the degree p whose covariant derivatives vanish. Then we have a relation*

$$B_P \geqq B'_P.$$

We shall remark first that the proof of this theorem by H. Iwamoto, depending on the theorem of de Rham [2] may be simplified if we use the following theorem of Hodge [3].

The p-th Betti number of an orientable compact positive definite Riemannian space is equal to the maximum number of linearly independent harmonic tensors of the degree p. A tensor $\xi_{a_1 \cdots a_p}$ is said to be harmonic if (1) it is skew-symmetric, and (2) it satisfies the conditions

$$\text{(A)} \qquad \xi_{a_1 \cdots a_p ; r} = \sum_{q=1}^{p} \xi_{a_1 \cdots a_{q-1} r a_{q+1} \cdots a_p ; a_q}$$

and

$$\text{(B)} \qquad \xi_{a_1 \cdots a_p ; r} g^{a_p r} = 0,$$

where the semi-colon denotes the covariant derivative with regard to the Christoffel symbols.

By this Hodge's therem, we can prove the above theorem as follows. Let $\xi_{a_1 \cdots a_p}$ be a skew-symmetric tensor whose covariant derivative vanishes. Then $\xi_{a_1 \ldots a_p}$ satisfies evidently the conditions (A) and (B).

Hence it becomes a harmonic tensor. If there exist two linearly independent skew-symmetric tensors $\xi_{a_1 \ldots a_p}$ and $\eta_{a_1 \cdots a_p}$ whose covariant derivatives vanish, then their linear combinations are skew-symmetric and their covariant derivatives vanish also. Hence they are also harmonic. Thus we have