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Riemann Spaces of Class Two and their Algebraic Characterization.

Part I.

Makoto MATSUMOTO.

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We shall investigate in this paper a necessary and sufficient condition that an n-dimensional Riemann space $R_n (n \geqq 6)$ be of class two. Let the line element of R_n be a positive definite quadratic form

$$ds^2 = g_{ij} dx^i dx^j \; ; \quad (i, j, \ldots = 1, 2, \ldots, n) \; ;$$

where g's are analytic functions of $x^1, \ldots, x^n$.

Consider, in an $(n+2)$-dimensional euclidean space E_{n+2}, an n-dimensional variety S_n defined by

$$y^a = \varphi^a (x^1, \ldots x^n) \qquad (a = 1, \ldots, \; n+2) \; ;$$

where y's are current coordinates of the point of S_n refered to a rectangular cartesian coordinate system in E_{n+2} and φ's are analytic functions of $x^1, \ldots, x^n$. The line element along a curve on S_n is given by

$$ds^2 = \sum_a (dy^a)^2 = \sum_a \; B_i^{\cdot a} \; B_j^{\cdot a} \; dx^i dx^j = g_{ij} \; dx^i dx^j \; ;$$

where

$$B_i^{\cdot a} = \frac{\partial y^a}{\partial x^i}.$$

Let $B_P^{\cdot a} (P = I, II)$ be the components of two mutually orthogonal unit vectors normal to S_n. The variation of $B_\lambda^{\cdot a} (a = 1, \ldots, n+2 \; ; \; \lambda = 1, \ldots n, \; I, \; II)$ along the curve can be written as

$$dB_\lambda^{\cdot a} = H_{\lambda i}^{\sigma} \; B_\sigma^{\cdot a} \; dx^i \quad (i = 1, \ldots, \; n \; ; \; \sigma, \lambda = 1, \ldots, \; n, \; I, II \; ; \; a = 1, \ldots, n+2).$$

As a condition of integrability of these equations we get immediately that $H_{jk}^{i} \; (i, j, k = 1, \ldots, n)$ are Christoffel's symbols and $H_{ij}^{P} \; (P = I, II \; ; \; i, j = 1, \ldots, n)$ are symmetric in i and j; and $H_{Qi}^{P} (P, Q = I, II \; ; \; i = 1, \ldots, n)$ are skew-symmetric in P and Q; those $H_{\lambda i}^{\sigma}$ satisfy the *Gauss equation*

$$(1) \qquad R_{ijkl} = H_{ik}^{P} \; H_{jl}^{P} - H_{il}^{P} \; H_{jk}^{P},$$