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On the Stability of the linear Transformation in Banach Spaces.

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Let E and E' be Banach spaces, and $f(x)$ be a transformation from E into E', which is "*approximately linear*". Ulam proposed the problem: *When does a linear transformation near an "approximately linear" transformation exist?* This was solved by D. H. Hyers[1]. The object of this paper is to generalize Hyer's theorem.

In generalizing the definition of Hyers, we shall call a transfor mation $f(x)$ from E into E' "*approximately linear*", when there exists $K(\geqq 0)$ and $p(0\leqq p<1)$ such that

$$\|f(x+y)-f(x)-f(y)\|\leqq K(\|x\|^{p}+\|y\|^{p})$$

for any x and y in E.

Let $f(x)$ and $\varphi(x)$ be transformations from E into E'. These are called "*near*", when there exists $K(\geqq 0)$ and $p(0\leqq p>1)$ such that

$$\|f(x)-\varphi(x)\|\leqq K\|x\|^{p}$$

for any x in E.

Theorem. *If $f(x)$ is an approximately linear transformation from E into E', then there is a linear transformation $\varphi(x)$ near $f(x)$. And such $\varphi(x)$ is unique.*

Proof. By the assumption there are $K_0(\geqq 0)$ and $p(0\leqq p<1)$ such that

$$\|f(2x)/2-f(x)\|\leqq K_0\|x\|^{p}. \tag{1}$$

We shall now prove that

$$\|f(2^{n}x)/2^{n}-f(x)\|\leqq K_0\|x\|^{p}\sum_{i=0}^{n-1}2^{i(p-1)} \tag{2}$$

for any integer n. The case $n=1$ holds by (1). Assuming the case

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1) D. H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci., vol 27, No. 4 (1941), p. 222-4.