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Change of variables in the multiple Lebesgue integrals.

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Rademacher's theorem[1] on the change of variables in the multiple Lebesgue integrals, though very important, is not found in any book on the theory of functions of real variables, so that I will give a simple proof of it in the following lines.

Let D be a domain in the $(x_1,\ldots\ldots,x_n)$-space and Δ be one in the $(u_1,\ldots\ldots,u_n)$ -space and D be mapped on Δ topologically by

$$T: \qquad u_i=f_i(x_1,\ldots\ldots,x_n), \tag{1}$$

$$T^{-1}: \qquad x_i=\varphi_i(u_1,\ldots\ldots, u_n), \quad (i=1,2,\ldots\ldots,n),$$

where f_i and φ_i are continuous in D and Δ respectively.

If any measurable set in D is mapped on a measurable set in Δ, then T is called a measurable mapping. It is well known[2] that the necessary and sufficient condition that T is a measurable mapping is that any null set[3] in D is mapped on a null set in Δ.

Theorem 1.[4] *If at every point* $(x_1,\ldots\ldots,x_n) \in D$,

$$\varlimsup_{h_1^2+\ldots+h_n^2\to 0} \frac{|f_i(x_1+h_1,\ldots,x_n+h_n)-f_i(x_1,\ldots,x_n)|}{\sqrt{h_1^2+\ldots+h_n^2}}$$

$$=L_i(x_1,\ldots,x_n)<\infty \qquad (i=1,2,\ldots\ldots,n), \tag{2}$$

then T *is a measurable mapping.*

Proof. We define a set A_k $(k=1,2,\ldots)$ of points $(x_1,\ldots,x_n)$ by the condition

(1) Rademacher: Über die partielle und totale Differentierbarkeit von Funktionen mehererer Veränderlichen und über die Transformation der Doppelintegrale. Math. Ann. **79**.

(2) Rademacher: Eineindeutige Abbildung und Messbarkeit. Monathefte f. Math. u. Phys. **27** (1916). Carathéodory: Vorlesungen über reelle Funktionen. p. 354.

(3) A set is called a null set, if its Lebesgue measure is zero.

(4) Rademacher. 1. c. (1), (2).