Journal of the Mathematical Society of Japan Vol. 2, Nos. 1–2, Sept., 1950.

On topological completeness

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(Received May 30, 1949)

E. Čech has proved the following theorem[1]:
A metrizable space R is topologically complete if and only if it is completely metrizable.

In this paper we shall show that by making use of the theorem of N. A. Shanin,[2] we can simplify the proof of Čech's theorem and generalize it slightly.

We mean in this paper by a filter a family of closed sets having the finite intersection property, and we say that a filter $\{F_\alpha \mid A\}$ is vanishing when $\Pi\, F_\alpha = \phi$ holds.

N. A. Shanin's theorem. *In order that a T_1-space R can be represented as an intersection of at most $\mathfrak{n}$ (a cardinal number) open sets in Wallman's bicompactification $W(R)$ of R, it is necessary and sufficient that there exists a collection $\{\mathfrak{F}_\tau\}$ of at most $\mathfrak{n}$ vanishing filters $\mathfrak{F}_\tau$ of R with the property: For an arbitrary maximum vanishing filter $\mathfrak{F}$ of R, there exists a filter $\mathfrak{F}_\tau$ of $\{\mathfrak{F}_\tau\}$ such that $\mathfrak{F}_\tau \subset \mathfrak{F}$.*

When we note that there exists a one-to-one correspondence between an open set of $W(R)$ containing R and a vanishing filter of R as well as between a point of $W(R)-R$ and a maximum vanishing filter of R, this theorem is almost obvious.

Proof of Čech's theorem. We begin with the necessity of the condition. Let R be a topologically complete and metrizable space. Since R is topologically complete, R is, as is well known, a G_δ-set in Čech's bicompactification $\beta(R)$ of R, *i.e.* an intersection of at most countable open sets of $\beta(R)$. Since R is metrizable, and accordingly normal, $\beta(R)$ and $w(R)$ are, as is well-known, identical. Therefore, when we use Shanin's theorem in the case of $\mathfrak{n}=\mathfrak{a}$, we get the family $\{\mathfrak{F}_n\}$ of at most a countable number of vaning filters $\mathfrak{F}_n$ mentioned in the theorem.
Let $\mathfrak{F}_n = \{F_{n,\alpha} \mid \alpha \in A_n\}$; then $\{F^c_{n,\alpha} \mid \alpha \in A_n\} = \mathfrak{M}_n$[3] $(n=1,2,\dots)$ are open coverings of R.

On the other hand, since R is metrizable, R has a base $\{\mathfrak{N}_m\}$ of uniform-