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On the structure and representations of Clifford algebras.

Yukiyosi KAWADA and Nagayosi IWAHORI.

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The structure and representations of Clifford algebras over the complex number field were studied by many authors.[1] The purpose of this note is to investigate them over any ground field K with $\chi(K)\neq 2$. Moreover, to apply the results to the problems of Eddington on sets of anticommuting matrices,[2] we shall consider slightly generalized Clifford algebras. In Appendix we shall give irreducible representations of such algebras in their explicite form.

1. Let K be any field with the characteristic $\chi(K)\neq 2$, and n, g two integers such that $0\leqq g\leqq n$, $n>0$. The *Clifford algebra of type* (n, g) $C(n,g)/K$ over K is defined as an algebra with generators.

$$u_0, u_1, \dots, u_n$$

and with fundamental relations

$$(1)\quad u_0^2=u_0,\ u_0u_i=u_iu_0=u_i,\ u_i^2=u_0\ (1\leqq i\leqq g),\ u_i^2=-u_0\ (g+1\leqq i\leqq n),$$
$$u_iu_j+u_ju_i=0\ \ (i\neq j,\ i>0,\ j>0).$$

$C(n,g)$ has rank 2^n and

$$u_0, u_i\ \ (1\leqq i\leqq n),\ u_iu_j\ \ (1\leqq i<j\leqq n), \dots\dots, u_1u_2\dots n_n$$

form a basis of $C(n,g)/K$.[3] $C(n,0)/K$ is the ordinary Clifford algebra.[3]

We distinguish now three cases according to the properties of K:

Case I. There is an element $\lambda\in K$ with $1+\lambda^2=0$.

Case II. There is no solution $\lambda\in K$ of $1+\lambda^2=0$, but there are elements

$$\alpha, \beta\in K \text{ with } 1+\alpha^2+\beta^2=0.$$

Case III. There are no solutions $\alpha, \beta\in K$ of $1+\alpha^2+\beta^2=0$.

All three cases may arise, when $\chi(K)=0$. Of course we have Case I when K is the complex number field, and Case III when K is the real number field. If $\chi(K)=p\neq 0$, then we have either Case I or Case II.[4] Case I occurs when $p\equiv 1$ (mod. 4), and Case II when $p\equiv 3$ (mod. 4) for prime field K.[5]

Now we consider three algebras. The one is the quarternion algebra $Q/K=C(2,0)/K$: