Journal of the Mathematical Society of Japan Vol. 2, Nos. 1-2, Sept., 1950.

On the Cluster Sets of Analytic Functions in a Jordan Domain.

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(Received Oct. 11, 1948)

I. Cluster Sets defined by the convergence set.

1. Let D be a Jordan domain, C its boundary, E any set on $D+C$(1) and z_0, z_0' two points on C. Divide C into two parts C_1 and C_2 by z_0 and z_0'. We denote the part of D, C, E, C_1 and C_2 in $|z-z_0|\leqq r$ by D_r, C_r, E_r, $C_r^{(1)}$ and $C_r^{(2)}$ respectively and the part of $|z-z_0|=r$ in D by θ_r. Let $w=f(z)$ be a meromorphic fnnction in D and $\mathfrak{D}_r$ the set of values taken by $f(z)$ in D_r. Then the intersection $\bigcap\limits_{r>0}\overline{\mathfrak{D}_r}=S_{z_0}^{(D)}$(2) is called the *cluster set* of $f(z)$ in D at z_0 and the intersection $\bigcap\limits_{r>0}\mathfrak{D}_r=R_{z_0}^{(D)}$ the *range of values* of $f(z)$ in D at z_0. The intersection $\bigcap\limits_{r>0}\overline{M_r^{(E)}}=S_{z_0}^{(E)}$, where $M_r^{(E)}$ is the union $\bigcup S_{z'}^{(D)}$, for $z_0\neq z'\in E$, $S_{z'}^{(D)}$ consisting of the single value $f(z')$ for $z'\in D$, is called the cluster set of $f(z)$ on E at z_0. For example, $S_{z_0}^{(C)}$, $S_{z_0}^{(C_1)}$, $S_{z_0}^{(C_2)}$ and $S_{z_0}^{(L)}$, where L is a Jordan curve in D terminating at z_0, are thus defined. If $S_{z_0}^{(L)}$ consists of only one value α, we call α the asymptotic value, L the asymptotic path and we denote the set of all the asymptotic values at z_0 by $\Gamma_{z_0}^{(D)}$, and call it the *convergence set* of $f(z)$ at z_0. When $f(z)$ omits at least three values in the neighbourhood of z_0(3), $\Gamma_{z_0}^{(D)}$ consists of at most one value (4). Then we call the value of non-empty $\Gamma_{z_0}^{(D)}$ the *boundary value* at z_0, and denote it by $f(z_0)$. Furthermore the intersection $\bigcap\limits_{r>0}\overline{Y_r^{(E)}}=\Gamma_{z_0}^{(E)}$ for $E\subset C$, $Y_r^{(E)}$ being the union $\bigcup\Gamma_{z'}^{(D)}$ for $z_0\neq z'\in E_r$, is called the cluster set of the convergence set of $f(z)$ on E at z_0.

$S_{z_0}^{(D)}$ includes all the other cluster sets and $S_{z_0}^{(E)}$ includes $\Gamma_{z_0}^{(E)}$. $S_{z_0}^{(D)}$, $S_{z_0}^{(C_1)}$, $S_{z_0}^{(C_2)}$ and $S_{z_0}^{(L)}$ are continuums but not necessarily $\Gamma_{z_0}^{(C)}$, $\Gamma_{z_0}^{(C_1)}$ and $\Gamma_{z_0}^{(C_2)}$ are (5).

2. Let $f(z)$ be bounded in the neighbourhood of z_0. Then it is known that (6)

$$\overline{\lim_{z\to z_0}}|f(z)|=\overline{\lim_{C\ni z'\to z_0}}\ (\overline{\lim_{z\to z'\neq z_0}}|f(z)|),$$

and that this is equivalent to $B(S_{z_0}^{(D)})\subset B(S_{z_0}^{(C)})$, $B(S)$ being the boundary set of S(7). Also it is known that $B(S_{z_0}^{(D)})\subset B(\Gamma_{z_0}^{(C)})$ holds in the case where D is a circle (8); then it holds also in the general case where D is a Jordan domain, by means of a one-to-one continuous corresponden-