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Note on the Cluster Sets of Analytic Functions.

Kiyoshi Noshiro.

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1. Let D be an arbitrary connected domain and C be its boundary. Let E be a closed set of capacity[1] zero, included in C and z_0 be a point in E. Suppose that $W=f(z)$ is a single-valued function meromorphic in D. We associate with z_0 three cluster sets $S_{z_0}^{(D)}$, $S_{z_0}^{(C)}$ and $S_{z_0}^{*(C)}$ as follows: $S_{z_0}^{(D)}$ is the set of all values a such that $\lim_{\nu\to\infty} f(z_\nu)=a$ with a sequence $\{z_\nu\}$ of points tending to z_0 inside D. $S_{z_0}^{*(C)}$ is the intersection $\bigcap_r M_r$, where M_r denotes the closure of the union $\bigcup_{z'} S_{z'}^{(D)}$ for all z' belonging to the common part of $C-E$ and $U(z_0, r)$: $|z-z_0|<r$. In the particular case when E consists of a single point z_0, we denote $S_{z_0}^{*(C)}$ by $S_{z_0}^{(C)}$ for the sake of simplicity. Obviously $S_{z_0}^{(D)}$ and $S_{z_0}^{*(C)}$ are closed sets such that $S_{z_0}^{*(C)}\subset S_{z_0}^{(D)}$, and $S_{z_0}^{(D)}$ is always non-empty while $S_{z_0}^{*(C)}$ becomes empty if and only if there exists a positive number r such that $C-E$ and $U(z_0, r)$ have no point in common.

Concerning the cluster sets $S_{z_0}^{(D)}$, $S_{z_0}^{(C)}$ and $S_{z_0}^{*(C)}$ the following theorems are known:

Theorem I. (Iversen-Beurling-Kunugi)[2] *$B(S_{z_0}^{(D)})\subset S_{z_0}^{(C)}$, where $B(S_{z_0}^{(D)})$ denotes the boundary of $S_{z_0}^{(D)}$, or, what is the same, $\Omega=S_{z_0}^{(D)}-S_{z_0}^{(C)}$ is an open set.*

Theorem II. (Beurling-Kunugi)[3] *Suppose that $\Omega=S_{z_0}^{(D)}-S_{z_0}^{(C)}$ is not empty and denote by Ω_n any connected component of Ω. Then $w=f(z)$ takes every value, with two possible exceptions, belonging to Ω_n infinitely often in any neighbourhood of z_0.*

Theorem. I* (Tsuji)[4] *$B(S_{z_0}^{(D)})\subset S_{z_0}^{*(C)}$, that is, $\Omega=S_{z_0}^{(D)}-S_{z_0}^{*(C)}$ is an open set.*

Theorem II*. (Kametani-Tsuji)[5] *Suppose that $\Omega=S_{z_0}^{(D)}-S_{z_0}^{*(C)}$ is not empty. Then $w=f(z)$ takes every value, except a possible set of w-values of capacity zero, belonging to Ω infinitely often in any neighbourhood of z_0.*

Evidently Theorem I* is a complete extension of Theorem I. It seems however that there exists a large gap between Theorem II and Theorem II*. The object of the present note is to show that under the assumption that *D is simply connected,* Theorem II* can be written in the form of Theorem II.