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An Asymptotic Series for the Number of Three-Line Latin Rectangles

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Introduction

An (n,k)-latin rectangle is a (k,n)-matrix having k permutations of degree n as its k rows and admitting no coincidences of letters in each of letters in each of its n columns. For the number $f(n,k)$ of such latin rectangles, P. Erdös and I. Kaplansky[1] recently proved an asymptotic relation

$$f(n,k)\sim e^{-k(k-1)/2}(n!)^{k}.$$

And for the special ease of $f(n, 3)$, numerous results are reported to be obtained by authors of the United States and other countries thongh we have access to only a few of them.[2]

In this paper we shall give some formulas for the number $f(n, 3)$. Explicit formulas are given in **1.** They would require heavy computations. Our principal aim is an asymptotic series for $f(n, 3)$.

$$f(n, 3)\sim e^{-3}(n!)^{3}\left\{1-\frac{1}{n}-\frac{1}{2}\cdot\frac{1}{n(n-1)}+\frac{5}{6}\cdot\frac{1}{n(n-1)(n-2)}+\frac{1}{24}\cdot\frac{1}{n(n-1)(n-2)(n-3)}-\dots\right\}$$

given in **2.** The close-up to the coefficients M_s of this series will be found in **3**, and finally in **4** numerical values of $N_s=s!\,M_s$ and $\psi_n=f(n, 3)/n!$ are given for $n\leqq 20$. Our series seems to give far better approximations than we can prove, at least as far as $n\leqq 20$.

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1. Explicit Formulas

We shall first modify the numbers $f(n, 2)$ and $f(n, 3)$ slightly:

$$\varphi_n=f(n, 2)/n!,\quad \psi_n=f(n, 3)/n!,$$

and use them exclusively. These are the numbers of *reduced* latin rectangles, i.e. of those latin rectangles, whose first rows consist of natural permutations. For φ_n a well-known theorem (*problème des rencontres*) states that