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A Theorem on compact semi-simple groups

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Let G be a topological group. We shall denote by $D(G)$ the closure of the commutator subgroup of G, and call a connected compact group *semi-simple* if $D(G)$ coincides with G. A connected compact Lie group is semi-simple in our sense if and only if it is a semi-simple Lie group. We note here the fact that any factor group of a connected compact semi-simple group is also semi-simple. In the present note we shall prove the following.

Theorem. *Let G be a connected compact semi-simple group. Then for any element x of G there corresponds a pair of elements y and z such that*

$$x=y^{-1}z^{-1}yz.$$

Similar results have been obtained by K. Shoda[1] for the special linear group over an algebraically closed field, and recently by H. Tôyama[2] for some types of compact simple Lie groups. Our theorem is an extension of the theorem of Tôyama.

In order to prove our theorem we shall first prove a special case, namely the following

Lemma. *Let G be a connected compact semi-simple Lie group. Then any element x in G is representable in a form $y^{-1}z^{-1}yz$ for suitably chosen elements y and z.*

Proof Let A be a maximal connected commutative subgroup of G. Then A is a closed toroidal group,[3] and any element of G is known to be conjugate with some element of A. Hence it is sufficient to prove the case when x is contained in A.

Now we introduce a system of canonical coordinates in A. Then any a of A is given by its coordinates:

$$a=a(\varphi), \quad \varphi=(\varphi_1,\dots\dots, \varphi_n).$$

where φ_i varies over all real numbers mod. 1. Let now H be the normalizer of A in G. The transformation by an element h of H induces a

1) K. Shoda: Einige Saetze ueber Matritzen, Jap. Journ. of Math. v. XIII, 1937.

2) H. Tôyama: On commutators of matrices.

3) See L. Pontrjagin: Topological groups, Princeton, 1939.