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On the harmonic prolongation.

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In the customary proofs of the theorems of the harmonic prolongation, it is usual to make use of Green's formula and so it is necessary to assume the functions to have the first derivatives continuous on a part or whole of the bonndary. In this note we prove them without using Green's formula and show that the theorems hold without the assumption of the continuity of the first derivatives.

Theorem. I. *Let $v(x,y)$ be harmonic in the domain, $(x-a)^2+y^2<r^2$, $y>0$, and continuous on its closure, $(x-a)^2+y^2\leqq r^2$, $y\geqq 0$, and suppose that its normal derivative vanishes on the real axis, $y=0$, $a-r<x<a+r$. If we define the function $v^*(x,y)$ by setting*

$$(1)\qquad v^*(x,y)=v(x,y) \quad \text{when } (x-a)^2+y^2\leqq r^2,\ y\geqq 0,$$

$$v^*(x,y)=v(x,-y) \quad \text{when } (x-a)^2+y^2\leqq r,\ y\leqq 0,$$

then $v^*(x,y)$ is a harmonic prolongation of $v(x,y)$ in the circle $(x-a)^2+y^2<r^2$.

Without loss of generality we can assume that $r=1$ and $a=0$. We consider the Poisson integral $V(x,y)$ for the boundary value of $v^*(x,y)$,

$$V(x,y)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(\varphi-\theta)+r^2}\,v^*(\xi,\eta)\,d\varphi,$$

where $\xi+i\eta=e^{i\varphi}$ and $x+iy=re^{i\theta}$, $0\leqq r<1$. $V(x,y)$ is harmonic in $x^2+y^2<1$, continuous on $x^2+y^2\leqq 1$, and its normal derivative vanishes on the real axis, $y=0$, $-1<x<1$, because of the continuity and symmetry of $v^*(x,y)$. Then $u(x,y)=V(x,y)-v(x,y)$ has the same properties as $v(x,y)$ and vanishes on the peripherie $x^2+y^2=1$, $y\geqq 0$. Hence if we can prove the following lemma, we obtain the theorem I.

Lemma. *If $u(x,y)$ is harmonic in the domain $x^2+y^2<1$, $y>0$ and continuous on its closure $x^2+y^2\leqq 1$, $y\geqq 0$ and satisfies the following boundary conditions*:

$$(2)\qquad u(x,y)=0 \qquad \text{for } x^2+y^2=1,\ y\geqq 0;$$