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On the faithful representations of Lie groups

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It is well known that any Lie algebra may be represented faithfully by matrices. This result, which was first established by I. Ado, was by his powerful method proved by E. Cartan.[2] K. Iwasawa[3] has recently given purely algebraic proof; His result is most general in the sense that he proved the theorem for Lie algebras over an arbitrary field. But Cartan's analytical method of proof is a most direct one and has an advantage enabling us to discuss simultaneously the problem in the large. In the present paper we first attempt to simplify the Cartan's proof. Namely we construct a Lie algebra (which we shall call normal) containing the given algebra and having a more simple structure than the given one. Hence the problem is reduced to that of the faithful representations of normal Lie algebras and this may readily be reduced to nilpotent case by Cartan's method. Thus we may proceed without the rather complicated arguments in solvable and general cases. Further we prove some results on the faithful representations of Lie groups in the large. Some of them were already obtained by E. Cartan[4] and A. Malcev.[5]

§ 1. Let L be a Lie algebra over a field P. A linear mapping d on L will be called a derivation if

$$d[x, y]=[dx, y]+[x, dy].$$

The mapping $d_a: x \rightarrow [a, x]$ is, as we may readily verify, a derivation which we shall call inner derivation defined by an elemeut $a \in L$.

In the following we assume that P is of characteristic 0.

Lemma 1.[6] Let d be a derivation of L. Then we may represent d uniquely in the form

$$(1) \qquad d=d^o+d^s, \qquad d^o d^s=d^s d^o,$$

1) Ado [1]
2) Cartan [3]
3) Iwasawa [1]
4) Cartan [6]
5) Malcev [3]. Malcev's papers [2] and [3] are not yet accessible to the writer. I knew his results on reading Mathematical Reveiews. Some were also obtained in M. Gotô [2] independently.
(6 Gantmacher [1]