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On Integral Invariants and Betti Numbers of Symmetric Riemannian Manifolds, II.

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Chapter III.

Formation of invariant differentials and the Schubert varieties.

I.

1. We have already seen that the manifold $S(n)$ can be considered as the set of all *null-systems* $x \to y = Sx$ such that the skew matrix S is at the same time orthogonal. Let us show that this manifold can also be considered as the set of all isotropic m-planes in P_n.

In order that a subspace $\mathfrak{M} \epsilon P$ be isotropic, it is neccesary and sufficient that $(x, y) = 0$ for all $x, y \epsilon \mathfrak{M}$. For any $\mathfrak{M}$ there exists the conjugate $\overline{\mathfrak{M}}$ of $\mathfrak{M}$. $\overline{\mathfrak{M}}$ is namely the set of vectors $\overline{x}$, where $x \epsilon \mathfrak{M}$. The correspondence $\mathfrak{M} \to \overline{\mathfrak{M}}$ is invariant under the group $O(n)$. In $\mathfrak{M}$ we take m vectors $x_1, \ldots\ldots, x_m$ such that $(x_i \; \overline{x}_j) = \delta_{ij}$. The vectors $e_1, \ldots\ldots, e_n$ with

$$(1) \qquad e_i = (x_i + \overline{x}_i) / \sqrt{2} \;, \quad e_{m+i} = (x_i - \overline{x}_i) / \sqrt{-2}$$

constitute a real coordinate system such that $(e_i \; e_j) = \delta$ (ij). This shows at once that the manifold $\sum(n)$ is tronsformed transitively by the group $OL(n.)$ Now the manifold $\sum(n)$ consists of two different continuous families, each being transformed transitively by the group $O(n)$. We denote one of them by $\sum(n)$. Then there exists a one to one correspondence between the elements of the manifolds $S(n)$ and $\sum(n)$ invariant with respect to the group $O(n)$. In fact, let S be an element of $S(n)$. We consider a set $\mathfrak{M}$ of all complex vectors of the form

$$\mathfrak{x} = x + \sqrt{-1}\, Sx \qquad x \epsilon R_n \,.$$

The vector x is isotropic. To show that $\mathfrak{M}$ is isotropic m-dimensional we take a special coordinate system such that $S = I$. $\mathfrak{M}$ is then the set of all vectors of the form $(z_1, \ldots\ldots, z_m, \sqrt{-1}\, z_1, \ldots\ldots, \sqrt{-1}\, z_m)$, where $z_i \epsilon K$.

The group of desplacements of $S(n)$ is $S \to TS^*T$, $T \epsilon O(n)$. We denote by $\Sigma(n)$ the set of all isotropic m-planes of P_n, where $n = 2m$.