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A Theorem on Riemann Sum.
Notes on Fourier Analysis (XIII)

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Let us consider the series

$$f(x)=\sum_{n=1}^{\infty}\frac{\cos nx}{n^{a}} \quad \left(0<a<\frac{1}{2}\right),$$

and let $f_n(x)$ be its n-th Riemann sum, i.e.

$$f_n(x)=\frac{1}{n}\sum_{k=1}^{n} f\left(x+2\pi\frac{k}{n}\right).$$

Since $f(x)$ is of order $1/x^{1-a}$ in the neighbourhood of the origin, we have

$$\lim_{n\to\infty}\sup f_n(x)=\infty$$

for almost all x, by the theorem due to J. Marcinkiewicz, A. Zygmund[1] and H. Ursell[2]

Connected with this fact it may be of some interest to prove the following

Theorem. *Let $f(x)$ be a function integrable in $(0,2\pi)$ and of period 2π. Let $f_n(x)$ be its Riemann sum and its Fourier series be*

(1) $$f(x)\sim\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx).$$

If the Fourier coefficients satisfy the condition

(2) $$\lim_{n\to\infty}\sum_{\nu=1}^{\infty}\left(|a_{n\nu}-a_{n(\nu+1)}|+|b_{n\nu}-b_{n(\nu+1)}|\right)=0,$$

in particular if

$$\sum_{n=1}^{\infty}\left(|a_n-a_{n+1}|+|b_n-b_{n+1}|\right)<\infty,$$

or if $\{a_n\}$, $\{b_n\}$ are non-increasing sequences, then for almost all x there exists a sequence of integers $\{m_k\}$ (depending on x) such that

$$\lim_{k\to\infty}f_{m_k}(x)=\int_0^{2\pi}f(x)\,dx$$

1) Mean values of trigonometrical polynomials, Fund. Math., **28** (1937), p. 131—166, spec., p. 157.

2) On the behaviour of a certain sequence of functions derived from a given one, Jour. London Math. Soc., **12** (1937).