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On the invariant differential forms of local Lie groups.

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Let **G** be an n-dimensional connected compact Lie group and let p_i $(i=0,1,\dots, n)$ be its Betti numbers. The polynomial $P_G(t)=\sum_{i=0}^{n} p_{n-i}t^i$ is called the *Poincaré polynomial* of **G.** H. Hopf [1] has proved in 1941 the following remarkable results:

(i) $P_G(t)=(1+t^{m_1})(1+t^{m_2})\dots(1+t^{m_l})$,

(ii) $m_i\equiv 1 \pmod{2}$ $(i=1,2,\dots, l)$,

(iii) $n=m_1+\dots\dots+m_l$.

On the other hand H. Cartan [1] has called $P_G(t)=\sum_{i=0}^{n} q_{n-i}t^i$ the Poincaré polynomial of **G** where q_i is the dimension of the module of all the i-th invariant differntial forms of **G** over the field of real numbers. Then $P_G(t)$ is defined also for a local Lie group. By the results of de Rham [1] and Cartan [1] $p_i=q_i$ $(i=0,1,\dots,n)$ hold for connected compact Lie groups.

In this note we shall prove that the Hopf's results (i), (ii) are also true for the Poincaré polynomials in Cartan's sense if we consider local Lie groups with the property **P**, formulated in Theorem 1. This property **P** concerns with the complete reducibility of some representations of **G.** Hence every compact Lie groups and every real seml-simple Lie groups have the property **P,** and we can apply our results for these Lie groups. Our proof rests entirely on the Cartan's local method of differential forms and does not use any topological methods in the large.

1. Let $\mathbf{G}=\{S_a\}$ be a real n-dimensional local Lie group with parameter $a=(a_1,\dots,a_n)$ in a neighbourhood U^n of $(0,\dots,0)$ in n-dimensional Euclid space R^n. Denote by $c_i=\varphi^i\ (a, b)$ $(i=1,\dots,n)$ the composition function of $\mathbf{G}$: $S_c=S_aS_b$. Let

$$\omega^p=\sum_{i(1)<\dots<i(p)} A_{i(1)\dots i(p)}(x)\ dx_{i(1)}\dots dx_{i(p)} \tag{1}$$

be a Grassmann-Cartan's differential form of degree p defined on some neighbourhood of $(0,\dots\dots,0)$ in R^n. The left (right) translations $x_i\to\bar{x}_i=\varphi_i(a, x)$ $(x_i\to\bar{\bar{x}}_i=\varphi_i(x, a))$ $(i=1,\dots,n)$ with parameter a induce the left (right) transformations