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On the cohomology theory of rings.

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Recently G. Hochschild has developed the theory of cohomology groups of associative algebras[1]. We shall consider in this paper some problems concerning the cohomology groups of rings. Especially we shall be able to characterize the vanishing of $\mathbf{H}_n(\mathbf{R}, \mathbf{m})$ for every **R-R**-module **m** in the case $n=1$ and 3 by the extension properties (Theorem 6 and 8).

In § 1 necessary definitions from Hochschild's theory are given. § 2 concerns the extensions of **R-R**-modules. In § 3 we define a useful mapping $F_{\beta\gamma}$ of $\mathbf{H}_n(\mathbf{R}, \mathbf{m})$ to $\mathbf{H}_{n+1}(\mathbf{R}, \mathbf{n})$ for any **R-R**-modules **m** and **n**, which is a generalization of the fundamental isomorphism of Hochschild. In § 4 we consider the special extension problem, which corresponds to the Teichmüller's theory for simple algebras[2]. These considerations can also be applied to Lie algebras, as I shall show in another paper.

§ 1. *Definitions of the cohomology groups of rings.*

Let **R** be a ring and **m** an **R-R**-module. Namely, we suppose that am, mb ($m \in \mathbf{m}$, $a, b \in \mathbf{R}$) belong to **m**, are linear, distributive in a, b, m, and satisfy the associative law $a(bm)=(ab)m$, $(ma)b=m(ab)$, $(am)b=a(mb)$. We call an element $f_0 \in \mathbf{m}$ a *0-cochain*, and $f_n(a_1, \ldots\ldots, a_n) \in \mathbf{m}$ ($a_i \in \mathbf{R}$), which is linear with respect to $a_1, \ldots\ldots, a_n$, a *n-cochain* ($n \geqq 1$). We denote the module of all n-cochains by $\mathbf{L}_n(\mathbf{R}, \mathbf{m})$. Moreover, we define the *coboundary operator* $\delta f_n = f_{n+1}$ ($f_n \in \mathbf{L}_n(\mathbf{R}, \mathbf{m})$, $f_{n+1} \in \mathbf{L}_{n+1}(\mathbf{R}, \mathbf{m})$) by

$$(\delta f_n)(a_1, \ldots\ldots, a_{n+1}) = a_1 f_n(a_2, \ldots\ldots, a_{n+1}) + \sum_{k=1}^{n} (-1)^k f_n(a_1, \ldots\ldots, a_k a_{k+1}, \ldots\ldots, a_{n+1}) + (-1)^{n+1} f_n(a_1, \ldots\ldots, a_n) a_{n+1}. \quad (1)$$

Then δ is a linear mapping and satisfies the relation $\delta(\delta f_n)=0$ for any f_n. We call an element f_n with $\delta f_n=0$ an *n-cocycle* ($n \geqq 0$) and an element f_n with $f_n = \delta g_{n-1}$ ($n \geqq 1$) an *n-coboundary*. We denote the module of all n-cocycles (n-coboundaries) by $\mathbf{C}_n(\mathbf{R}, \mathbf{m})$ ($\mathbf{B}_n(\mathbf{R}, \mathbf{m})$). And we define the *n-cohomology group* $\mathbf{H}_n(\mathbf{R}, \mathbf{m}) = \mathbf{C}_n(\mathbf{R}, \mathbf{m})/\mathbf{B}_n(\mathbf{R}, \mathbf{m})$ ($n \geqq 1$).

§ 2. *Extension of* **R-R**-*module and* 1-*cohomology group* $\mathbf{H}_1(\mathbf{R}, \mathbf{m})$.

Def. Let **m**, **n** be two **R-R**-modules. We call an another **R-R**-module **M** an *extension of* **m** *by* **n**, if (i) $\mathbf{M} \supseteqq \mathbf{n}$, (ii) $\mathbf{M}/\mathbf{n} \cong \mathbf{m}$ (as **R-R**-module), (iii)