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Note on faithful modular representations of a finite group.

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In a recent note[1] the writer has studied the structure of finite groups possessing a faithful irreducible representation (i.r.), directly indecomposable representation (d.i.r.), or a faithful directly indecomposable component (d.i.c.) of the regular representation (r.r.) in a modular field of characteristic $p \neq 0$ (p-modular field). The result is similar to the case of groups with faithful non-modular i.r.[2] Namely: Let $\mathfrak{M}$ be the product of the totality of minimal abelian invariant subgroups of order prime to p in a finite group $\mathfrak{G}$, and let $\mathfrak{M}=\mathfrak{L}_1 \times \mathfrak{L}_2 \times \ldots\ldots \times \mathfrak{L}_g$ be its decomposition into subgroups of prime power orders with different primes $l_i (\neq p)$. *$\mathfrak{G}$ possesses a faithful p-modular i.r., if and only if $\mathfrak{G}$ has no invariant subgroup $\neq 1$ whose order is a power of p and moreover the following condition is satisfied:*

(*) *every $\mathfrak{L}_i$ possesses an invariant subgroup with cyclic factor group which contains no invariant subgroup $\neq 1$ of $\mathfrak{G}$.*

$\mathfrak{G}$ has a faithful d.i.c. of p-modular r.r. (or a faithful p-modular d.i.r. whatsoever), if and only if the condition (*) is satisfied.

(Furthermore, (*) is equivalent to that

(†) each $\mathfrak{L}=\mathfrak{L}_i$ is a product of c, say, mutually $\mathfrak{G}$-isomorphic minimal invariant subgroups of $\mathfrak{G}$ and the inequality $c \leq m/\lambda$ is satisfied, where l^m is the order of the minimal factor and l^λ is the number of elements in the $\mathfrak{G}$-automorphism quasifield of the minimal factor.)

As a corollary of the result we have: 1) *If $\mathfrak{G}$ has a faithful non-modular i.r. then it has a faithful d.i.c. of p-modular r.r. (for any p)*; 2) *If $\mathfrak{G}$ possesses faithful p-modular and q-modular d.i.r. with distinct p, q, then it has a faithful non-modular i.r.*

The present note is to supplement these by giving mutual relations between such modular and non-modular representations. We prove namely[3]:

I. *If a group[4] $\mathfrak{G}$ possesses a faithful non-modular i.r.[5] $M(\mathfrak{G})$, then any d.i.c. $V(\mathfrak{G})$ of a modular r.r. containing $M(\mathfrak{G})$, in the sense of R. Brauer-C. Nesbitt[6], is faithful.*

II. *If $\mathfrak{G}$ possesses a faithful non-modular i.r.[7], then an arbitrary faithful d.i.c. of a modular r.r. contains a faithful non-modular i.r.*