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On linearly ordered groups.

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A group G is called a *linearly ordered group* (=l. o. group), when in G is defined a linear order $a>b$, preserved under the group multiplication:

$$a>b \text{ implies } ac>bc \text{ and } ca>cb \text{ for all } c \text{ in } G.$$

A typical example is the additive group R of all real numbers with respect to the usual order. Subgroups of R are also linearly ordered and they are, as is well known, characterized among other l. o. groups by the condition that their linear orders are archimedean, that is to say, that for any positive elements[1] a, b there is a positive integer n, so that it holds

$$a^n>b, \ b^n>a.$$

Everett and Ulam have proved that we can define a linear order in a free group with two generators, so that it becomes a l. o. group[2]. In the following we shall generalize this theorem in the form that any l. o. group can be obtained by an order homomorphism from a proper l. o. free group, and then study the general character of group- and order-structure of these groups. Finally we shall add some examples which will illustrate our theorems.

We prove first some lemmas.

Lemma 1. Let G be a l. o. group and P the set of all positive elements in G. P has then following properties:

i) *$e \notin P$, and if $x \neq e$ either $x \in P$ or $x^{-1} \in P$.*

ii) *$x \in P$ and $y \in P$ implies $xy \in P$.*

iii) *if $x \in P$, then $axa^{-1} \in P$ for all a in G.*

Conversely, if a group G contains a subset P, having the properties i), ii), iii), we can then introduce a linear order in G by defining,

$$a>b, \text{ if } ab^{-1} \in P. \tag{1}$$

Proof. The former part is almost obviovs. We have only to note that iii) follows from $axa^{-1}>aea^{-1}=e$ for $x>e$. We prove the latter part According to i) and $ba^{-1}=(ab^{-1})^{-1}$ it can be seen that one and only one of the relations