COMPACT SET IN UNIFORM SPACE AND FUNCTIONS SPACES*)

BY

HISAHARU UMEGAKI

The purpose of this paper is to discuss some compactness problems in uniform space and in a space of continuous functions whose domain and range are both uniform spaces. It is known that a uniform structure in uniform space may be represented by a family of pseudo-metrics. Using this we shall prove a convex linear topological space can be imbedded into a direct product of normed spaces (§ 1). We shall next prove a compactness theorem of the Kolmogoroff-Tulajkov type in uniform space (§ 2). We introduce furthermore some topologies into the space of continuous functions and prove a compactness theorem of the Ascoli-Arzelà type, and as an application we shall prove similar theorem for character group of topological group.

§ 1. **On the uniform structure.** Let E be a uniform space defined by the uniform structure $\{V_\alpha\}_{\alpha\in\mathfrak{A}}$. After A. Weil, for each V_α we shall define pseudo-metric d_α such that:

$$d_\alpha(p, q) \geqq 0, \ d_\alpha(p, p) = 0, \ d_\alpha(p, q) \leqq d_\alpha(p, r) + d_\alpha(r, q)$$

and $p = q \longleftrightarrow d_\alpha(p, q) = 0$ for all $\alpha\in\mathfrak{A}$.

We define a structure by $W_{\alpha\varepsilon} = \{(p, q); d_\alpha(p, q) < \varepsilon\}$; then $\{W_{\alpha\varepsilon}\}_{\alpha,\ \varepsilon > 0}$ is equivalent to $\{V_\alpha\}$. We can replace the triangle condition of $\{d_\alpha\}$ by the following: for each $\alpha\in\mathfrak{A}$, there exists $\beta_\alpha = \beta\in\mathfrak{A}$ such that $d_\alpha(p, q) \leqq d_\beta(p, r) + d_\beta(r, q)$.

The same consideration can be applied for linear topological space (l. t. s.).

Let L be a l. t. s. defined by the neighbourhood (nbd.) system $\{U_\alpha\}_{\alpha\in\mathfrak{A}}$ of the origin θ. D. H. Hyers [1] has proved that there exists a family of pseudo-norms $\{|\cdot|_\alpha\}$ satisfying the following conditions:

(a) for every $x\in L$ and $\alpha\in\mathfrak{A}$, $|x|_\alpha \geqq 0$.

(b) for every real λ, $x\in L$ and $\alpha\in\mathfrak{A}$, $|\lambda x|_\alpha = |\lambda| \cdot |x|_\alpha$.

(c) for every $\alpha\in\mathfrak{A}$, there exists $\beta_\alpha = \beta\in\mathfrak{A}$ such that $|x+y|_\alpha \leqq |x|_\beta + |y|_\beta$ for all $x, y\in L$.

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