NOTES ON FOURIER ANALYSIS (XLIV): ON THE SUMMATION OF FOURIER SERIES

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This paper consists of three parts. In § 1, we investigate the relation between the logarithmic order of a certain Cesàro mean of function and that of Cesàro sum of its Fourier series (Theorem 1. 2). Our theorem is best possible in a sense (Theorem 1. 3) and incidentally it is proved that there is an integrable function which is (C, α) continuous and whose Fourier series is not (C, α) summable. In § 2 we prove a theorem generalizing F. T. Wang's [5] (Theorem 2. 1). We give further a relation between Riesz continuity of function and Cesàro summability of its Fourier series (Theorem 2. 3). In § 3 we prove a theorem concerning absolute Riesz summability with its converse (Theorem 3. 1).

1. We suppose that $\varphi(t)$ is an even integrable function, with period 2π. We denote the α-th integral of $\varphi(t)$ by

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1}\varphi(u)\,du \qquad (\alpha > 0), \tag{1.1}$$

and the α-th mean of $\varphi(t)$ by

$$\varphi_\alpha(t) = \Gamma(\alpha+1)\, t^{-\alpha}\Phi_\alpha(t). \tag{1.2}$$

Let us write the Fourier series of $\varphi(t)$ as

$$\mathfrak{S}[\varphi] = \sum_{n=0}^{\infty} a_n \cos nt$$

and denote its (C, α)-mean by $C_\alpha(\omega)$.

By the result of Bosanquet [2, pp. 26–27], we have

$$C_\alpha(\omega) = \omega \int_0^\pi \varphi_\alpha(t) J_\alpha^\alpha(\omega t)\,dt + o(1), \qquad \text{as } \omega \to \infty, \tag{1.3}$$

where $J_\alpha^\alpha(u)$ satisfies the relation

$$|J_\alpha^\alpha(u)| \leqq K \tag{1.4}$$

for $u \geqq 0$, and

$$J_\alpha^\alpha(u) = (\sin(u - \pi\alpha))/u + O(1/u^2) \tag{1.5}$$

for large u.

We have then the following theorem:

THEOREM 1. 1. *If*

$$\int_0^t |\varphi_\alpha(t)|\,dt = o(t(\log 1/t)^r), \qquad (\alpha > 0,\ -1 < r < \infty), \tag{1.6}$$