# A DEFORMATION THEOREM ON CONFORMAL MAPPING

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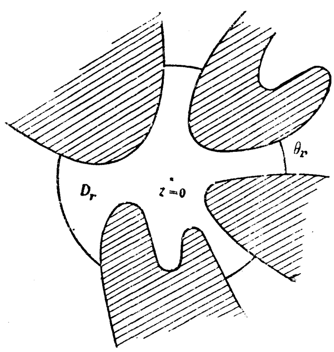
**1.** Let $D$ be a simply connected domain on $z$-plane, which contains $z=0$ and $z=\infty$ belongs to its boundary. The boundary $\Gamma$ of $D$ consists of at most a countable number of curves $C$, which extend to infinity in the both directions. Let $z_0\,(|z_0|=r_0)$ be the point on $\Gamma$, which lies nearest to $z=0$, then a circle $|z|=r(>r_0)$ meets $D$ in a number of cross cuts. We consider only such cross cuts, which separate $z=0$ from $z=\infty$ in $D$ and denote them by $\theta_r^{(i)}(i=1,2,\cdots, \ n=n(r))$.

We assume that $n(r)$ is finite, but may tend to infinity for $r\to\infty$. We put $\theta_r=\sum_{i=1}^{n}\theta_r^{(i)}$ and $r\theta(r)$ be the total length of $\theta_r$. $\theta_r$ divides $D$ into $n+1$ simply connected domains. Let $D_r$ be the simply connected one, which contains $z=0$, then $D_r$ is bounded by $\theta_r$ and a part of $\Gamma$.




We will prove the following theorem, which is a generalization of Ahlfors' deformation theorem.

THEOREM 1. *If we map $D$ conformally on $|w|<1$ by $w=f(z)$ $(f(0)=0)$, then the image of $\theta_r$ in $|w|<1$ can be enclosed in a finite number of circles $K_r^{(i)}$ $(i=1,2,\cdots, \ \nu(r)\leqq n(r))$, which cut $|w|=1$ orthogonally, such that the sum of radii is less than*

$$\text{const.}\ \exp\left(-\pi\int_{r_0}^{kr}\frac{dr}{r\theta(r)}\right), \qquad (0<k<1),$$

*where $k$ is any positive number less than* 1

When $D$ is bounded by only one curve, then by Ahlfors' deformation theorem, we can prove easily that we can take $k=1$. Hence our theorem is worse than Ahlfors' deformation theorem, but is more general, since $D$ may be bounded by a countable number of curves.

PROOF. First we map $D$ conformally on $\Im\zeta>0$ by $\zeta=\varphi(z)$ $(\varphi(z_0)=\infty,\ \varphi(0)=i)$, then $z=\infty$ is mapped on a bounded closed set $E$ of masure zero on $\Im\zeta=0$. Let $\lambda_r^{(i)}$ be the image of $\theta_r^{(i)}$ then $\lambda_r^{(i)}$ is a Jordan arc, whose both end points lie on $\Im\zeta=0$. Let $\Delta_r^{(i)}$ be the finite domain,